Calculus I

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Lecture No. 1

Functions

1.1 What is a function?

> **Definition 1.** Function. A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The set D is caled the domain of the funciton. The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

In other words, $\{x \in D | f(x) \in E\}$.

1.1.1 Ways to represent a function

There are many different ways of describing funcitons:

- verbally (by a description of words)
- numerically (by a table of values)
- visually (by using a graph)
- algebraically (by using an explicit formula)

1.1.2 Domain and Range

Domain is the x-span and **range** is the y-span. Find the domains of the following functions:

Exercise 1.1.1.

$$f(x) = \frac{1}{x^2 - 5x + 6}.$$

Exercise 1.1.2.

$$g(x) = \frac{1}{1 - cos^2 x}.$$

Exercise 1.1.3.

$$h(x) = \frac{1}{\sqrt{x-1}} - \sqrt{1-x}.$$

Solution 1.1.1.

$$x^2 - 5x + 6 = (x-3)(x-2) \quad \therefore \{x \in \mathbb{R} | x \neq 3, 2\}.$$

Solution 1.1.2.

$$cos^2(x) \neq 1 \quad \therefore cos(x) \neq 1, -1 \quad \therefore \{x \in \mathbb{R} | x \neq n\pi, n \in \mathbb{N}\}.$$

Solution 1.1.3.

$$x > 1, \quad x \leqslant 1 \quad \therefore \text{No solutions for } x.$$

1.1.3 Vertical line rule

A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.
Example 1.1.1. The parabola $x = y^2 - 2$ is not the graph of a function of x because there are vertical lines that intersect the parabola twice.

1.1.4 Piecewise defined functions

Definition 2. Piecewise defined functions. Piecewise defined functions are functions defined by different formulas in different parts of their domains.

Example 1.1.2. The following is a piecewise defined function:

$$f(x) = \begin{cases} 1-x, & x \leqslant -1 \\ x^2, & x > -1 \end{cases}$$

1.1.5 Even & Odd Functions

Even and odd functions are used to define whether a function is symmetrical or perfectly asymmetrical about the line $x = 0$.

Definition 3. Even function. An even function is a function that satisfies the following:

$$f(-x) = f(x).$$

for all x in its domain. Even functions show symmetry about the line $x = 0$.

Example 1.1.3. The function $cos(x)$ is symmetric about the line $x = 0$ and $cos(-x) = cos(x)$ is true, therefore, $cos(x)$ is an even function.

Definition 4. Odd function. An odd function is a function that satisfies the following:

$$f(-x) = -f(x).$$

for all x in its domain. Odd functions show perfect asymmetry about the line $x = 0$.

Example 1.1.4. The function $sin(x)$ is perfectly asymmetric about the line $x = 0$ and $sin(-x) = -sin(x)$ is true, therefore, $sin(x)$ is an odd function.

Determine whether each of the following functions is even, odd, or neither even nor odd:

Exercise 1.1.4.

$$f(x) = x^5 + x.$$

Exercise 1.1.5.

$$g(x) = 1 - x^4.$$

Exercise 1.1.6.

$$h(x) = 2x - x^2.$$

Solution 1.1.4.

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \\ &= -f(x). \end{aligned}$$

Therefore, f is an odd function.

Solution 1.1.5.

$$\begin{aligned} g(-x) &= 1 - (-x)^4 \\ &= 1 - x^4 \\ &= g(x). \end{aligned}$$

Therefore, g is an even function.

Solution 1.1.6.

$$\begin{aligned} h(-x) &= 2(-x) - (-x)^2 \\ &= -2x - x^2 \end{aligned}$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Proof 1. Prove that a function f is the sum of an even and an odd function in the range $(-a, a)$.

$$\begin{aligned} f(x) &= e(x) + o(x) \\ f(-x) &= e(-x) + o(-x) \\ f(-x) &= e(x) + o(-x) \\ f(-x) &= e(x) - o(x) \end{aligned}$$

1.1.6 Increasing & Decreasing Functions

Definition 5. Increasing Function. A function f is called increasing on an interval I if:

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

The inequality $f(x_1) < f(x_2)$ must be satisfied for every pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

The opposite is true for decreasing functions.

1.2 A catalog of essential functions

Definition 6. Polynomial. A function P is called a polynomial if it is of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0.$$

where n is a non-negative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

Definition 7. Power Functions. A function of the form:

$$f(x) = x^a.$$

where a is a constant, is called a power function.

Definition 8. Rational Functions. A rational function f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}.$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$.

Definition 9. Algebraic functions. A function is called an algebraic function if it can be constructed using algebraic operations starting with polynomials.

Definition 10. Exponential Functions. The exponential functions are the functions of the form $f(x) = b^x$, where the base b is a positive constant.

1.3 Transformation of Functions

Suppose $c > 0$. To obtain the graph of:

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward.
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward.
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right.
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left.
- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.
- $y = \frac{1}{c}f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c.
- $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c.
- $y = f(\frac{x}{c})$, stretch the graph of $y = f(x)$ horizontally by a factor of c.
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

1.4 Combinations of Functions

Definition 11. Sum and difference functions. The sum and difference functions are defined by:

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x).$$

If the domain of f is A and the domain of g is B, then the domain of $f \pm g$ is the intersection $A \cap B$ because both $f(x)$ an $g(x)$ have to be defined.

Definition 12. Product and quotient functions. The product and quotient functions are defined by:

$$(fg)(x) = f(x)g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The domain of fg is $A \cap B$. Because we cannot divide by 0, the domain of $\frac{f}{g}$ is therefore $\{x \in A \cap B | g(x) \neq 0\}$.

Definition 13. Composite functions. Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by:

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f:

$$Dom(f \circ g) = \{x \in Dom(g) | g(x) \in Dom(f)\}.$$

Lecture No. 2

Limits

2.1 Intuition

Here is how we denote a limit:

$$\lim_{x \to a} f(x).$$

read as the limit of the function $f(x)$ as x approaches a. Think about the function:

$$f(x) = \frac{\sin(x)}{x}.$$

Focus on how it looks close to the point $x = 0$. Even though $f(x) = \frac{\sin(x)}{x}$ has no value defined at the point $x = 0$, we can still look at the values f takes at points cles to $x = 0$, and see if those values are approaching some fixed value as x gets closer and closer to 0. The limit of a function $f(x)$ at a point a is a number L which the values of the function get closer and closer to as the variable x gets closer and closer to a.

We know that:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Definition 14. Tangent line to a graph. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Then we define the **tangent line** to the graph of $f(x)$ at the point x_0 to be the line going through the point $(x_0, f(x_0))$ with gradient:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

if the limit exists. If this limit does not exist then we say that the graph does not have a tangent at that point.

Example 2.1.1. Suppose that a ball is dropped from a tower 450m above the ground. Find the velocity of the ball after 5 seconds.
Solution 2.1.1. We are dealing with a single instant of time ($t = 5$), so no time interval is involved. We can approximate the desired quantity by computing the

average velocity over the brief time interval τ seconds starting at time $t = 5$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5+\tau) - s(5)}{\tau} \\ &= \frac{4.9(5+\tau) - 4.9 \times 5^2}{\tau} \\ &= 49 + 4.9\tau \end{aligned}$$

As the time interval τ gets smaller and smaller, this quantity will be a better approximation fo the instantaneous velocity at time τ. The instantaneous velocity after 5 seconds is $v = 49m/s$.

2.2 Limit Laws

We use theorems called limit laws to build up more complicated limits from simpler pieces.

Theorem 1. Sum Law. Let f and g be functions and let $a \in \mathbb{R}$. If the limits:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x).$$

exist, then $\lim_{x \to a}(f(x) + g(x))$ exists and is:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

The limit of a sum of functions is the sum of the limits of those functions.

Theorem 2. Product Law. Let f and g be functions and let $a \in \mathbb{R}$. If the limits:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x).$$

exist, then

$$\lim_{x \to a} f(x)g(x).$$

exists and

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x).$$

The limit of a product of functions is the product of the limits of those functions.

Theorem 3. Quotient Law. Let f and g be functions and let $a \in \mathbb{R}$. If the limits:

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x).$$

exist and

$$\lim_{x\to a} g(x) \neq 0.$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

The limit of a quotient of functions is the quotient of the limits of those functions.

There are also two simple functions whose limits we can assume.

Theorem 4. Limits of simple functions.
Let $a, c \in \mathbb{R}$, then:

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x = a$$

Theorem 5. Constant Multiple Law. Let f be a function, and let $a \in \mathbb{R}$. Suppose that c is a constant. If the limit:

$$\lim_{x\to a} f(x).$$

exist, then

$$\lim_{x\to a} cf(x).$$

exists and

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x).$$

The limit of a constant times a function is the constant times the limit of the function.

If we use the product law in theorem 2 repeatedly with $g(x) = f(x)$, we obtain the following power law:

Theorem 6. Power Law. Let f be a function and let $a \in \mathbb{R}$. Suppose that n is a positive integer. If the limit:

$$\lim_{x \to a} f(x).$$

exists, then

$$\lim_{x \to a} [f(x)]^n.$$

exists, and

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n.$$

And lastly,

Theorem 7. Root Law. Let f be a function and let $a \in \mathbb{R}$. Suppose that n is a positive integer. If the limit:

$$\lim_{x \to a} f(x).$$

exists, then

$$\lim_{x \to a} \sqrt[n]{f(x)}.$$

exists, and

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}.$$

If n is even, we assume that $\lim_{x \to a} f(x) > 0$.

Example 2.2.1. Does the following limit exist, and if it does, evaluate it:

$$\lim_{x \to 3} (x^2 - 9).$$

We see that:

$$\begin{aligned}
\lim_{x \to 3} (x^2 - 9) &= \lim_{x \to 3} x^2 + \lim_{x \to 3} (-9) \\
&= \lim_{x \to 3} x \cdot \lim_{x \to 3} x + \lim_{x \to 3} (-9) \\
&= 3 \times 3 + (-9) \\
&= 0.
\end{aligned}$$

Therefore, the limit exists and is equal to 0.

Theorem 8. Direct Substitution Property. If f is a polynomial or a rational function and a is in the domain of f, then:

$$\lim_{x \to a} = f(a).$$

Functions with this property are called *continuous.*

Example 2.2.2. Evaluate the limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Note that $\lim_{x\to 1}(x-1)=0$. This means we cannot use the quotient law in theorem 3, but must take an alternative approach:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1}(x+1)\\ &= 1+1\\ &= 2.\end{aligned}$$

This solution implicitly uses a very simple and important fact known as the replacement law.

Theorem 9. Replacement Law. Let f and g be functions and let $a\in\mathbb{R}$. Assume that $f(x)=g(x)$ whenever $x\neq a$. Then either both f and g have limits at a and:

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

or, neither limit exists.

Example 2.2.3. Evaluate the limit:

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}.$$

Unable to apply quotient law in theorem 3, since limit of the denominator is 0. We can rationalise the numerator (called multiplying by the conjugate):

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\\ &= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)}\\ &= \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}\end{aligned}$$

Using the quotient law in theorem 3:

$$= \frac{\lim_{t\to 0}1}{\lim_{t\to 0}\left(\sqrt{t^2+9}+3\right)}$$

Using the sum law in theorem 1:

$$= \frac{1}{\lim_{t\to 0}\sqrt{t^2+9}+\lim_{t\to 0}3}$$

Using the root law in theorem 7:

$$\begin{aligned}&= \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+\lim_{t\to 0}3}\\ &= \frac{1}{6}\end{aligned}$$

Some limits are best calculated by first finding the left and right hand limits.

Theorem 10. This statement holds true:

$$\lim_{x \to a} f(x) = L.$$

if and only if:

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

A two sided limit exists if and only if both of the one-sided limits exist and are equal.

Example 2.2.4. Prove that the following limit does not exist:

$$\lim_{x \to 0} \frac{|x|}{x}.$$

Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, approaching from the right:

$$\begin{aligned} \lim_{x \to 0^+} \frac{|x|}{x} &= \lim_{x \to 0^+} \frac{x}{x} \\ &= \lim_{x \to 0^+} 1 \\ &= 1 \end{aligned}$$

Approaching from the left:

$$\begin{aligned} \lim_{x \to 0^-} \frac{|x|}{x} &= \lim_{x \to 0^-} \frac{-x}{x} \\ &= \lim_{x \to 0^-} (-1) \\ &= -1 \end{aligned}$$

Since the right and left hand limits are different, if follows from theorem 10 that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

The next two theorems give two additional properties of limits.

Theorem 11. Let f and g be functions, and let $a \in \mathbb{R}$. Assume that:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x) \quad \text{exist.}$$

And that $f(x) \leqslant g(x)$ for all x in an open interval of the form $(a - \delta, a + \delta)$ for some positive $\delta \in \mathbb{R}$ (except possibly at a itself). Then:

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x).$$

2.3 The Squeeze Theorem

Theorem 12. Squeeze Theorem. Let f, g, and h be functions, and let $a \in \mathbb{R}$. Assume that:

$$\lim_{x\to a} f(x) \text{ exists.}$$

$$\lim_{x\to a} h(x) \text{ exists.}$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) \text{ is true.}$$

And that $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval of the form $(a-\delta, a+\delta)$ for some positive $\delta \in \mathbb{R}$ (except possibly at a itself).

Then:

$$\lim_{x\to a} g(x) \text{ exists.}$$

And:

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x) \text{ is true.}$$

Example 2.3.1. Show that:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Solution 2.3.1. Note that we cannot use:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}.$$

because $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. Instead, we apply the squeeze theorem. Note that the function in question is bounded by the $\sin\frac{1}{x}$ function as such:

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1.$$

The inequality remains true when multiplied by a positive number. Since we know that all values of x^2 are positive, we can do the following:

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2.$$

And we know that:

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} -x^2 = 0.$$

Therefore using the squeeze theorem we can conclude that:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

2.4 The Precise Definition of a Limit

Consider the function:

$$f(x) = \begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}.$$

It is clear that $\lim_{x\to 3} f(x) = 5$.

Now let us ask:

> How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x-3|$ and the distance from $f(x)$ to 5 is $|f(x)-5|$ so we need a number δ such that:

$$|f(x)-5| < 0.1 \quad \text{if} \quad |x-3| < \delta \quad \text{but } x \neq 3.$$

Note that if $|x-3| > 0$ then $x \neq 3$ so:

$$|f(x)-5| < 0.1 \quad \text{if} \quad 0 < |x-3| < \delta.$$

Notice that if $0 < |x-3| < (0.1)/2 = 0.05$, then:

$$\begin{aligned} |f(x)-5| &= |(2x-1)-5| \\ &= |2x-6| \\ &= 2|x-3| < 2(0.05) = 0.1 \end{aligned}$$

And therefore

$$|f(x)-5| < 0.1 \quad \text{if} \quad 0 < |x-3| < 0.05.$$

Note that we can generalise to:

$$|f(x)-5| < \varepsilon \quad \text{if} \quad 0 < |x-3| < \frac{\varepsilon}{2}.$$

Which can be rewritten to:

$$\text{if} \quad 3-\delta < x < 3+\delta \quad \text{then} \quad 5-\varepsilon < f(x) < 5+\varepsilon.$$

Which is a precise way of saying that $f(x)$ is close to 5 when x is close to 3, because we can make $f(x)$ within an arbitrary distance ε from 5 by restricting the values of x to be within a distance $\frac{\varepsilon}{2}$ from 3.

Definition 15. Precise definition of a Limit. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write:

$$\lim_{x\to a} f(x) = L.$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that:

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x)-L| < \varepsilon.$$

or in clearer terms:

$$\text{if} \quad a-\delta < x < a+\delta \quad \text{then} \quad L-\varepsilon < f(x) < L+\varepsilon.$$

Example 2.4.1. Prove that:

$$\lim_{x\to 3}(4x-5) = 7.$$

Step 1: Preliminary analysis of the problem (guessing a value for δ). Let ε be a given positive number. We want to find a number δ such that:

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |(4x-5)-7| < \varepsilon.$$

But we can see that:

$$\begin{aligned} |(4x-5)-7| &= |4x-12| \\ &= |4(x-3)| \\ &= 4|x-3| \end{aligned}$$

Therefore we want a δ such that:

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad 4|x-3| < \varepsilon.$$

Which therefore means that:

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x-3| < \frac{\varepsilon}{4}.$$

Which suggests that we should choose $\delta = \frac{\varepsilon}{4}$.

Step 2: Proof. Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$. If $0 < |x-3| < \delta$, then:

$$\begin{aligned} |(4x-5)-7| &= |4x-12| \\ &= 4|x-3| < 4\delta \\ 4\delta &= 4\left(\frac{\varepsilon}{4}\right) \\ &= \varepsilon. \end{aligned}$$

Thus:

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |(4x-5)-7| < \varepsilon.$$

Therefore, by the definition of the limit:

$$\lim_{x \to 3} (4x-5) = 7.$$

2.4.1 One Sided Limits

The intuitive definitions of one-sided limits can be precisely formulated as follows:

Definition 16. Left-Hand Limit. The following equation holds true:

$$\lim_{x \to a^-} f(x) = L.$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that:

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Definition 17. Right-Hand Limit. The following equation holds true:

$$\lim_{x \to a^+} f(x) = L.$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that:

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Example 2.4.2. Use the right hand limit definition to prove that $\lim_{x \to 0^+} \sqrt{x} = 0:$

First, guess a value for δ. Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that:

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon.$$

or:

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad \sqrt{x} < \varepsilon.$$

If we square both sides of the inequality $\sqrt{x} < \varepsilon$, we get:

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2.$$

Suggesting that we should choose $\delta = \varepsilon^2$. Now we need to show that this δ works.

Given that $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then:

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon.$$

so

$$|\sqrt{x} - 0| < \varepsilon.$$

According to the definition of the right hand limit, this shows that:

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

2.4.2 Infinite Limits

Infinite limits can also be defined in a precise way.

Definition 18. Infinite Limit. Let f be a function defined on some open interval that contains the number a, excepts possibly at a itself. Then:

$$\lim_{x \to a} f(x) = \infty.$$

means that for every positive number M there is a positive number δ such that:

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M.$$

Example 2.4.3. Use the definition of the infinite limit to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

First, let M be a given positive number. We want to find a number δ such that:

$$\text{if} \quad 0 < |x - 0| < \delta \quad \text{then} \quad \frac{1}{x^2} > M.$$

But:

$$\begin{aligned} \frac{1}{x^2} > M &\iff x^2 < \frac{1}{M} \\ &\iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \\ &\iff |x| < \frac{1}{\sqrt{M}} \end{aligned}$$

So if we choose $\delta = \frac{1}{\sqrt{M}}$ and $0 < |x| < \delta = \frac{1}{\sqrt{M}}$, then $\frac{1}{x^2} > M$. This shows that $\frac{1}{x^2} \to \infty$ as $x \to \infty$.

Similarly, the following is a precise definition of the negative infinite limit.

Definition 19. Let f be a function defined on some open interval that contains the number a, except possible at a itself. Then:

$$\lim_{x \to a} f(x) = -\infty.$$

means that for every negative number N there is a positive number δ such that:

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N.$$

Lecture No. 3

Continuity

3.1 Precise Definitions

Definition 20. Continuous. A function f is continuous at a number a if:

$$\lim_{x \to a} f(x) = f(a).$$

The definition implicitly requires three things if f is continuous at a :

1. $f(a)$ is defined (that is, a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

Definition 21. One Sided Continuity. A function f is continuous from the right at a number a if:

$$\lim_{x \to a^+} f(x) = f(a).$$

and f is continuous from the left at a if:

$$\lim_{x \to a^-} f(x) = f(a).$$

Definition 22. Continuous on an Interval. A function f is continuous on an interval if it is continuous at every number in the interval.

Example 3.1.1. Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$

Solution 3.1.1. If $-1 < a < 1$, then using the limit laws, we have:

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(1-\sqrt{1-x^2}) \\
&= 1 - \lim_{x\to a}\sqrt{1-x^2} \\
&= 1 - \sqrt{\lim_{x\to a}(1-x^2)} \\
&= 1 - \sqrt{1-a^2} \\
&= f(a)
\end{aligned}$$

Thus by the definition of continuity, f is continuous at a if $-1 < a < 1$. We can also see that:

$$\lim_{x\to 1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x\to 1^-} f(x) = 1 = f(1).$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore we can see that f is continuous on $[-1, 1]$.

3.2 Composition of Continuous Functions

If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a :

$$\begin{array}{ccc} f+g & f-g & cf \\ fg & \frac{f}{g} \text{ if } g(a) \neq 0 & \end{array}$$

It follows that if f and g are continuous on an interval, then so are the functions stated above.

Theorem 13. Direct Substitution Property. Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$. Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational Functions
- Root Functions
- Trigonometric Functions

Theorem 14. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then:

$$\lim_{x\to a} f(g(x)) = f(b).$$

In other words:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

3.3 Root Law

Let g be a function and let $a \in \mathbb{R}$. Suppose that n is a positive integer. If the limit $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} \sqrt[n]{g(x)}$ exists and:

$$\lim_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to a} g(x)}.$$

If n is even, we assume that $\lim_{x\to a} g(x) > 0$.

3.4 Intermediate Value Theorem

Theorem 15. Intermediate Value Theorem. If f is continuous on the interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.