Calculus II

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March 13, 2023

Contents

Lecture No. 1

Sequences

1.1 Notation & Representations

1.1.1 Notation

The sequence:

$$\{a_1, a_2, a_3, \dots\}.$$

is usually denoted by one of the following notations:

$$\{a_n\} \quad \{a_n\}_{n=1}^{\infty} \quad (a_n) \quad (a_n)_{n=1}^{\infty}.$$

Often times, the index of a sequence may start from an integer other than 0 or 1.

Note. A sequence of real numbers can be considered as a **function** from a subset of $\mathbb{N}$ to $\mathbb{R}$.

Example 1.1.1. Sequence defined by a formula for the n-th term:

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \implies a_n = \frac{n}{n+1}.$$

Therefore the sequence's **listing** is of the following form:

$$\left\{\underbrace{\frac{1}{2}}_{n=1}, \underbrace{\frac{2}{3}}_{n=2}, \underbrace{\frac{3}{4}}_{n=3}, \dots, \frac{n}{n+1}, \dots\right\}.$$

1.1.2 Missing start index

If the starting index of a sequence is not stated, then it will be the smallest integer that the formula defining a_n makes sense and is a real number.

Example 1.1.2. Determine the starting index for the sequence:

$$a_n = \sqrt{n-3}.$$

The term $a_n = \sqrt{n-3}$ does not make sense for $n - 3 < 0$. It is only a real number when $n - 3 \geqslant 0$. Thus we have:

$$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}.$$

Example 1.1.3. Determine the starting index for the sequence:

$$b_n = \ln(n^2 - 5).$$

The term is defined when $n^2 - 5 > 0$, and therefore:

$$n > \sqrt{5}, \quad n \in \mathbb{N} \implies n \geqslant 3.$$

Thus we have:

$$\left\{\ln(n^2 - 5)\right\}_{n=3}^{\infty}.$$

1.1.3 Recursively defined sequences

The Fibonacci sequence is defined recursively by the following conditions:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-3} \; (n \geqslant 3).$$

The formula for f_n involves the preceding terms f_{n-1} and f_{n-2}, making it recursive.

1.2 Convergent Sequences & their Limits

1.2.1 Convergent & Divergent Sequences

The sequence:

$$\left\{\frac{n}{n+1}\right\}.$$

approaches the number 1 as n becomes larger. Therefore we say the sequence **converges** to 1, and we write:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The sequence $\{-1, 1, -1, 1, \dots\}$ does not approach a number as n becomes larger. Therefore we say the sequence **diverges**. The sequence:

$$\left\{\frac{n^3+1}{2}\right\}.$$

becomes larger and larger **uboundedly** as n becomes larger. Therefore we say the sequence **diverges**, and we write:

$$\lim_{n\to\infty} \frac{n^3+1}{2} = \infty.$$

1.2.2 Formal Definition of the Limit of a Sequence

Definition 1. A sequence $\{a_n\}$ has the limit L and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

if for every $\epsilon > 0$ there is a corresponding integer N such that:

$$|a_n - L| < \epsilon \quad \text{for every} \quad n > N.$$

If such a real number L exists, we say that the sequence **converges** to L. Otherwise, we say that the sequence **diverges** and is **divergent**. To prove that the limit exists, we have to find (or show that it exists) a positive integer N such that:

$$n > N \implies |a_n - L| < \epsilon.$$

Example 1.2.1. Use the definition to prove that the sequence $\left\{\frac{1}{n}\right\}$ **converges** to 0.

Proof. Let $\epsilon > 0$ be an arbitrarily chosen positive number. Our aim is to find an integer N such that for all $n > N$,

$$\left|\frac{1}{n} - 0\right| < \epsilon \implies \left|\frac{1}{n}\right| < \epsilon \implies \frac{1}{n} < \epsilon.$$

Thus we have $n > \frac{1}{\epsilon}$. We choose N to be an integer such that $N > \frac{1}{\epsilon}$. For example, we let $N = \left\lceil 1 + \frac{1}{\epsilon} \right\rceil = 1 + \left\lceil \frac{1}{\epsilon} \right\rceil$. For $n > N$, we have:

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

By definition, we have proved that $\{\frac{1}{n}\}$ converges to 0:

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

□

Example 1.2.2. Suppose $0 < |r| < 1$. Prove that $\lim_{n\to\infty} r^n = 0$.

Proof. Let $\epsilon > 0$ be an arbitrary positive real number. Our aim is to find an integer N such that $\forall n > N$,

$$|r^n - 0| = |r|^n < \epsilon.$$

which is equivalent to:

$$n \ln |r| < \ln \epsilon.$$

Note. We can only do this because $\ln x$ is a **strictly increasing** function. Since $0 < |r| < 1$, the real number $\ln |r| < 0$. Thus we have:

$$n > \frac{\ln \epsilon}{\ln |r|}.$$

We choose:

$$N = \left\lceil 1 + \frac{\ln \epsilon}{\ln |r|} \right\rceil.$$

so that:

$$N > \frac{\ln \epsilon}{\ln |r|}.$$

Thus, $\forall n > N$ we have:

$$n > \frac{\ln \epsilon}{\ln |r|} \implies |r^n - 0| < \epsilon.$$

Proving that:

$$\lim_{n\to\infty} r^n = 0.$$

□

Example 1.2.3. Use the definition to prove that:

$$\lim_{n\to\infty} \frac{n^2}{n^2+n-3} = 1.$$

Proof. Let $\epsilon > 0$ be an arbitrary positive real number. Our aim is to find an integer N such that $\forall n > N$,

$$\left| \frac{n^2}{n^2+n-3} - 1 \right| < \epsilon.$$

Note that:

$$\left| \frac{n^2}{n^2+n-3} - 1 \right| = \left| \frac{-(n-3)}{n^2+n-3} \right| = \frac{n-3}{n^2+n-3}, \text{ for every } n > 3.$$

and that:

$$n > 3 \implies n - 3 < n.$$

and that:

$$n > 3 \implies n^2 + n - 3 > n^2 \implies \frac{1}{n^2+n-3} < \frac{1}{n^2}.$$

and therefore:

$$n > 3 \implies \frac{n-3}{n^2+n-3} < \frac{n}{n^2} = \frac{1}{n}.$$

Now, if both $n > 3$ and $n > \frac{1}{\epsilon}$, then we have:

$$\frac{n-3}{n^2+n-3} < \frac{1}{n} < \epsilon.$$

Therefore we choose N to be an integer such that $N > \max\left(3, \frac{1}{\epsilon}\right)$:

$$N = 1 + \left\lceil \max\left(3, \frac{1}{\epsilon}\right) \right\rceil.$$

Thus, $\forall n > N$ we have:

$$\left| \frac{n^2}{n^2+n-3} - 1 \right| < \epsilon.$$

This proves that:

$$\lim_{n\to\infty} \frac{n^2}{n^2+n-3} = 1.$$

□

Example 1.2.4. Use the definition of the limit of a sequence to prove that:

$$\lim_{n\to\infty} \frac{1}{n} = 1.$$

is **not** valid.

Proof. Note that:

$$\left|\frac{1}{n}-1\right| > \frac{1}{2}, \text{ if } n > 2.$$

Therefore, let $\epsilon = 0.5$, then there is **no** positive integer N such that:

$$n > N \implies \left|\frac{1}{n}-1\right| < 0.5.$$

Which concludes that:

$$\lim_{n\to\infty} \frac{1}{n} = 1.$$

is not valid. □

1.3 Properties of Convergent Sequences

1.3.1 Constant Sequences

The sequence $\{a_n = C\}$, where $a_n = C$ for every n and C is a real number is known as a constant sequence, and its limit is C :

$$\lim_{n\to\infty} a_n = C.$$

1.3.2 Uniqueness of the Limit

Theorem 1. Uniqueness. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L'$, then:

$$L = L'.$$

Proof. Uniqueness of a Limit. Suppose $L = L'$ is not true:

$$L = L' \implies L \neq L' \implies L - L' \neq 0.$$

Consider the case where in the formal definition of the limit, epsilon has the following value:

$$\epsilon = \frac{|L-L'|}{2}.$$

Which can be used since:

$$L - L' \neq 0 \implies \epsilon > 0.$$

We know by the definition of the limit that:

$$\lim_{n\to\infty} a_n = L \iff \exists N_1 \in \mathbb{N} \text{ such that } n > N_1 \implies |a_n - L| < \epsilon.$$

$$\lim_{n\to\infty} a_n = L' \iff \exists N_2 \in \mathbb{N} \text{ such that } n > N_2 \implies |a_n - L'| < \epsilon.$$

Which therefore implies that:

$$|a_n - L| < \frac{|L-L'|}{2} \tag{1.1}$$

$$|a_n - L'| < \frac{|L-L'|}{2} \tag{1.2}$$

Note that:

$$|L - L'| = |L - a_n + a_n - L'|.$$

And using the triangle inequality:

$$|L - a_n + a_n - L'| \leqslant |L - a_n| + |a_n - L'|.$$

Therefore:

$$|L - L'| \leqslant |L - a_n| + |a_n - L'|. \tag{1.3}$$

if $(n > N_1) \wedge (n > N_2)$ then by equations (1.1) to (1.2):

$$\begin{aligned} |a_n - L| + |a_n - L'| &< \frac{|L - L'|}{2} + \frac{|L - L'|}{2} \\ |a_n - L| + |a_n - L'| &< |L - L'| \end{aligned}$$

Then using equation (1.3):

$$|L - L'| \leqslant |a_n - L| + |a_n - L'| < |L - L'|.$$

Therefore:

$$|L - L'| < |L - L'|.$$

A strict inequality, which contradicts our assumptions, therefore the theorem is true. □

1.3.3 Bounded Sequences

A sequence is bounded if there exists a **positive** real number M such that:

$$|a_n| \leqslant M, \quad \forall n.$$

A sequence is said to be bounded from above if there is a real number M such that:

$$a_n \leqslant M, \quad \forall n.$$

and we say that M is an **upper bound** of $\{a_n\}$. A sequence is said to be bounded from below if there is a real number m such that:

$$m \leqslant a_n, \quad \forall n.$$

and we say that m is a **lower bound** of $\{a_n\}$.

Theorem 2. Bounded. If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded. Consequently, an unbounded sequence is divergent.

Proof. Every convergent sequence is bounded. We know by the definition of the limit of a convergent sequence that:

$$\lim_{n\to\infty} a_n = L \iff \forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t } n > N \implies |a_n - L| < \epsilon.$$

Take $\epsilon = 1$ then:

$$\exists N_1 \in \mathbb{N} \text{ s.t } n > N_1 \implies |a_n - L| < 1.$$

Note that because of the triangle inequality:

$$|a_n - L| \geqslant |a_n| - |L|.$$

So for $n > N_1$, we have:

$$|a_n| - |L| \leqslant |a_n - L| < 1.$$

Therefore:

$$|a_n| - |L| < 1.$$

And therefore:

$$|a_n| < |L| + 1.$$

Now, let $M_1 = \max(|a_1|, |a_2|, \ldots, |a_{N_1}|)$. This means that:

$$n \leqslant N_1 \implies |a_n| \leqslant M_1.$$

Now take $M = \max(M_1, (|L| + 1))$. Then:

$$|a_n| \leqslant M, \quad \forall n \in \mathbb{N}.$$

□

1.3.4 Sum & Difference Laws

Suppose $\{a_n\}$ and $\{b_n\}$ are **convergent**. Then $\{a_n + b_n\}$ is convergent and:

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof. Suppose the following:

$$\lim_{n\to\infty} a_n = A, \quad \lim_{n\to\infty} b_n = B, \quad A, B \in \mathbb{R}.$$

Let $\epsilon > 0$ then:

$$\lim_{n\to\infty} a_n = A \iff \exists N_1 \in \mathbb{N} \text{ s.t } n > N_1 \implies |a_n - A| < \frac{\epsilon}{2} \tag{1.4}$$

$$\lim_{n\to\infty} b_n = B \iff \exists N_2 \in \mathbb{N} \text{ s.t } n > N_2 \implies |b_n - B| < \frac{\epsilon}{2}. \tag{1.5}$$

Now let $N = \max(N_1, N_2)$ so that:

$$n > N \implies (n > N_1) \wedge (n > N_2).$$

and note that:

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)|.$$

by the triangle inequality:

$$|(a_n - A) + (b_n - B)| \leqslant |a_n - A| + |b_n - B|.$$

And by equations (1.4) to (1.5), for $n > N$:

$$|(a_n - A) + (B - b_n)| \leqslant |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

and finally:

$$n > N \implies |(a_n + b_n) - (A + B)| < \epsilon.$$

□

Suppose $\{a_n\}$ and $\{b_n\}$ are **convergent**. Then $\{a_n - b_n\}$ is convergent and:

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

Proof. Suppose the following:

$$\lim_{n\to\infty} a_n = A, \quad \lim_{n\to\infty} b_n = B, \quad A, B \in \mathbb{R}.$$

Let $\epsilon > 0$ then:

$$\lim_{n\to\infty} a_n = A \iff \exists N_1 \in \mathbb{N} \text{ s.t } n > N_1 \implies |a_n - A| < \frac{\epsilon}{2} \tag{1.6}$$
$$\lim_{n\to\infty} b_n = B \iff \exists N_2 \in \mathbb{N} \text{ s.t } n > N_2 \implies |b_n - B| < \frac{\epsilon}{2}. \tag{1.7}$$

Now let $N = \max(N_1, N_2)$ so that:

$$n > N \implies (n > N_1) \wedge (n > N_2).$$

and note that:

$$|(a_n - b_n) - (A - B)| = |(a_n - A) - (b_n - B)|.$$

by the triangle inequality:

$$|(a_n - A) - (b_n - B)| \leqslant |a_n - A| + |b_n - B|.$$

And by equations (1.6) to (1.7), for $n > N$:

$$|(a_n - A) + (B - b_n)| \leqslant |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

and finally:

$$n > N \implies |(a_n + b_n) - (A + B)| < \epsilon.$$

□

1.3.5 Order Properties for Convergent Sequences

Theorem 3. If $\{a_n\}$ is **convergent** and $a_n \geqslant 0$, $\forall n \geqslant N_0$, then:

$$\lim_{n\to\infty} a_n \geqslant 0.$$

Therefore:

$$L \geqslant 0.$$

Proof. Suppose $\{a_n\}$ is convergent and $a_n \geqslant 0$, $\forall n \geqslant N_0$. Now suppose the limit L opposes the property in theorem 3:

$$\neg(L \geqslant 0) \equiv L < 0.$$

Consider the case where $\epsilon = -\frac{L}{2}$ and since $L < 0$, this value of ϵ is valid. This means that:

$$\exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n - L| < -\frac{L}{2}.$$

and therefore:

$$-\left(-\frac{L}{2}\right) < a_n - L < -\frac{L}{2}.$$

The latter part of the inequality can be rearranged as such:

$$a_n - L < -\frac{L}{2} \to a_n < L - \frac{L}{2} \to a_n < \frac{L}{2}.$$

and by the supposition at the start of the proof:

$$L < 0 \implies \frac{L}{2} < 0.$$

therefore:

$$a_n < 0.$$

which contradicts the assumptions held at the start of the proof. □

Theorem 4. If $\{a_n\}$ is **convergent** and $m \leqslant a_n \leqslant M$, $\forall n \geqslant N_0$, then:

$$m \leqslant \lim_{n\to\infty} a_n \leqslant M.$$

Proof. Suppose the following conditions:

$$\lim_{n\to\infty} a_n = L,\ L \in \mathbb{R} \qquad m \leqslant a_n \leqslant M,\ \forall n > N_0.$$

Now considering the sequence $\{a_n - m\}$ constructed using the laws in section 1.3.4, we see the following property:

$$\forall n > N_0,\ a_n - m \geqslant 0.$$

therefore by theorem 3:

$$\begin{aligned}
\lim_{n\to\infty}(a_n - m) \geqslant 0 &\equiv \lim_{n\to\infty} a_n - \lim_{n\to\infty} m \geqslant 0 && \text{by difference law}\\
&\equiv \lim_{n\to\infty} a_n - m \geqslant 0 && \text{by constant sequence}\\
&\equiv \lim_{n\to\infty} a_n \geqslant m
\end{aligned}$$

Now considering the sequence $\{M - a_n\}$ constructed using the laws in section 1.3.4, we see the following property:

$$\forall n > N_0,\ M - a_n \geqslant 0.$$

therefore by theorem 3:

$$\begin{aligned}
\lim_{n\to\infty}(M - a_n) \geqslant 0 &\equiv \lim_{n\to\infty} M - \lim_{n\to\infty} a_n \geqslant 0 && \text{by difference law}\\
&\equiv M - \lim_{n\to\infty} a_n \geqslant 0 && \text{by constant sequence}\\
&\equiv M \geqslant \lim_{n\to\infty} a_n
\end{aligned}$$

and therefore:

$$m \leqslant \lim_{n\to\infty} a_n \leqslant M.$$

□

Theorem 5. If $\{a_n\}$ and $\{b_n\}$ are **convergent** and $a_n \leqslant b_n, \forall n \geqslant N_0$, then:

$$\lim_{n\to\infty} a_n \leqslant \lim_{n\to\infty} b_n.$$

Proof. Suppose the following conditions:

$$\{a_n\},\ \{b_n\} \text{ are convergent} \qquad a_n \leqslant b_n,\ \forall n \geqslant N_0.$$

for $n \geqslant N_0$, we see that:

$$a_n \leqslant b_n \to b_n - a_n \geqslant 0.$$

Now considering the sequence $\{b_n - a_n\}$ constructed using the laws in section 1.3.4, and by theorem 3:

$$\begin{aligned}\lim_{n\to\infty}(b_n - a_n) \geqslant 0 &\equiv \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \geqslant 0\\ &\equiv \lim_{n\to\infty} b_n \geqslant \lim_{n\to\infty} a_n\end{aligned}$$

□

1.4 Divergent Sequences

To prove that a sequence $\{a_n\}$ is divergent, we have to show that for every arbitrary real number $L \in \mathbb{R}$, we can find a positive real number $\epsilon > 0$ such that for every positive integer N, there is some integer $n > N$ where $|a_n - L| \geqslant \epsilon$.

Example 1.4.1. Prove that the sequence $\{(-1)^n\}$ diverges.

Proof. Note that $a_{2n} = 1$ and $a_{2n+1} = -1$, so for an arbitrary real number L, either of the following conditions hold:

$$|a_{2n} - L| \geqslant \frac{1}{2} \text{ or } |a_{2n+1} - L| \geqslant \frac{1}{2}.$$

So taking $\epsilon = \frac{1}{2}$ for every positive integer N, we have either:

$$|a_{2N} - L| \geqslant \frac{1}{2} \text{ or } |a_{2N+1} - L| \geqslant \frac{1}{2}.$$

and there is always an $n > N$ such that $|a_n - L| \geqslant \frac{1}{2}$. Therefore, we conclude that the sequence diverges. □

1.4.1 Divergence to ∞

We say a sequence $\{a_n\}$ diverges to ∞, denoted $\lim_{n\to\infty} a_n = \infty$, if:

$$\forall M \in \mathbb{R}^+ > 0,\ \exists N \in \mathbb{N} \text{ s.t. } a_n > M,\ \forall n > N.$$

We say a sequence $\{a_n\}$ diverges to $-\infty$, denoted $\lim_{n\to\infty} a_n = -\infty$, if:

$$\forall M \in \mathbb{R}^+ > 0,\ \exists N \in \mathbb{N} \text{ s.t. } a_n < -M,\ \forall n > N.$$

Example 1.4.2. Prove by definition:

$$\lim_{n\to\infty} \frac{n^2}{n+1} = \infty.$$

Proof. For every positive real number M, we must find a positive integer N such that:

$$n > N \implies \frac{n^2}{n+1} - M > 0.$$

Note that:

$$\frac{n^2}{n+1} = (n-1) + \frac{1}{n+1}.$$

Given any positive real number M:

$$\frac{n^2}{n+1} - M = (n-1) + \frac{1}{n+1} - M.$$

we see that:

$$(n-1) + \frac{1}{n+1} - M > n - 1 - M > 0.$$

so we choose $N = M + 1$, which works for any M. Therefore we conclude that:

$$\lim_{n\to\infty} \frac{n^2}{n+1} = \infty.$$

$\square$

Lecture No. 2

Limit Evaluation

2.1 Properties of Limits

Theorem 6. Interchange Order. Suppose $\lim_{n\to\infty} a_n = L$ and f is continuous at L. Then:

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Example 2.1.1. Evaluate $\lim_{n\to\infty} \sin(\frac{1}{n})$.

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)\right) = \sin 0 = 0.$$

Example 2.1.2. Evaluate $\lim_{n\to\infty} e^{\frac{1}{n}}$.

$$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = e^0 = 1.$$

Example 2.1.3. Evaluate $\lim_{n\to\infty}\left(\frac{n^2}{n^2+3}\right)^3$.

$$\lim_{n\to\infty}\left(\frac{n^2}{n^2+3}\right)^3 = \left(\lim_{n\to\infty}\frac{n^2}{n^2+3}\right)^3 = 1^3 = 1.$$

Proof. Interchange Order. Let $\epsilon > 0$. Since $f(x)$ is continuous at $x = L$, $\exists\delta > 0$ such that:

$$|x - L| < \delta \implies |f(x) - f(L)| < \epsilon.$$

and this can be rewritten in context to:

$$|a_n - L| < \delta \implies |f(a_n) - f(L)| < \epsilon \tag{2.1}$$

Since $\lim_{n\to\infty} a_n = L$, $\exists N \in \mathbb{Z}$ such that:

$$\begin{aligned} n > N &\implies |a_n - L| < \delta \\ &\implies |f(a_n) - f(L)| < \epsilon \qquad \text{by equation (2.1)} \end{aligned}$$

And therefore we have proven that:

$$\lim_{n\to\infty} f(a_n) = f(L) = f\left(\lim_{n\to\infty} a_n\right).$$

□

2.2 Subsequences

Definition 2. Subsequence. A **subsequence** (denoted as $\{a_{n_k}\}_k$) of $\{a_n\}_n$ is a sequence obtained from the sequence a_n, such that:

$$n_1 < n_2 < n_3 \cdots .$$

Example 2.2.1. State the terms of the subsequence $\{a_{n^2}\}$:

$$\{a_{n^2}\} = a_1, a_4, a_9, a_{16}, \dots .$$

Example 2.2.2. State the terms of the subsequence $\{a_{n+3}\}$:

$$\{a_{n+3}\} = a_4, a_5, a_6, a_7, \dots .$$

Theorem 7. Convergence of subsequences. If $\lim_{n\to\infty} a_n = L$, then every subsequence of $\{a_n\}$ converges to L.

Example 2.2.3. We know that:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Therefore, by the convergence of subsequences, the following limits are true:

$$\lim_{n\to\infty} \frac{2n}{2n+1} = 1$$
$$\lim_{n\to\infty} \frac{5n+1}{(5n+1)+1} = 1$$
$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Proof. We want to prove that:

$$\lim_{n\to\infty} a_n = L \implies \lim_{n\to\infty} a_{n_k} = L.$$

Note. $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$.

We also know that since n_k is strictly increasing, that the following inequality must hold true:

$$k \leqslant n_k.$$

Now, let $\epsilon > 0$. We want to show that $\exists k_0 \in \mathbb{Z}$ such that:

$$|a_{n_k} - L| < \epsilon \;\forall k > k_0.$$

Since $\lim_{n\to\infty} a_n = L$:

$$\exists N \in \mathbb{Z} \text{ such that } n > N \implies |a_n - L| < \epsilon.$$

And since $n_k \geqslant k$, for $k > N$, we have $n_k > N$ and therefore:

$$n_k > N \implies |a_{n_k} - L| < \epsilon.$$

Proving that:

$$\lim_{n\to\infty} a_{n_k} = L.$$

□

As a consequence, we can conclude that $\lim_{n\to\infty} a_n$ does not exist in the following cases:

1. There is a divergent subsequence of $\{a_n\}$.

2. There are two convergent subsequences of $\{a_n\}$ with different limits.

Example 2.2.4. Consider the following sequence:

$$a_n = \begin{cases} \ln\left(\frac{1}{n}\right) & \text{if } n = 3k, \\ \pi & \text{if } n \neq 3k. \end{cases}$$

Here the subsequence $\{a_{3n}\} = \left\{\ln\left(\frac{1}{3n}\right)\right\}$. Thus it diverges, and therefore the sequence $\{a_n\}$ diverges.

Example 2.2.5. Consider the following sequence:

$$\{a_n\} = \{(-1)^n\}.$$

The odd subsequence $\{a_{2n-1}\} = \{-1\}$ while the even subsequence $\{a_{2n}\} = \{1\}$. Thus the sequence diverges.

Theorem 8. Convergence of odd and even subsequences to the same limit. When the odd subsequence and the even subsequence converge to the same limit, then the sequence converges to that same limit.

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1} = L \iff \lim_{n\to\infty} a_n = L.$$

Example 2.2.6. Determine the convergence of the following sequence:

$$a_n = \begin{cases} \frac{2k}{2k+1} & \text{if } n = 2k, \\ e^{1/(2k-1)} & \text{if } n = 2k-1. \end{cases}$$

Evaluating the even subsequence:

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right) = 1.$$

And evaluating the odd subsequence:

$$\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} e^{1/(2k-1)} = e^{\lim_{n\to\infty} \frac{1}{2n-1}} = e^0 = 1.$$

Since both the odd and even subsequences converge to 1, we can conclude:

$$\lim_{n\to\infty} a_n = 1.$$

2.3 Techniques

2.3.1 Dividing by the highest power of n in the denominator

When evaluating the limit of a sequence that is expressed as a fraction, we can divide both the numerator and the denominator by the highest power of n in the denominator:

Example 2.3.1. Determine if the following sequence converges, and if so, find its limit:

$$a_n = \frac{1+n}{2+n}.$$

Dividing by the highest power of n in the denominator:

$$\lim_{n\to\infty} \frac{1+n}{2+n} = \lim_{n\to\infty} \frac{1+n}{2+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{2}{n}+1} = \frac{0+1}{0+1} = 1.$$

Example 2.3.2. Determine if the following sequence converges, and if so, find its limit:

$$a_n = \frac{n^2+n+3}{5+n}.$$

Note that:

$$\lim_{n\to\infty} \frac{n^2+n+3}{5+n} = \lim_{n\to\infty} \frac{n^2+n+3}{5+n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{3}{n^2}}{\frac{5}{n^2}+\frac{1}{n}} = \infty.$$

And therefore we can conclude that the sequence diverges.

2.3.2 Multiplying by the conjugate

Certain limits can be rationalised by multiplying them by their conjugate:
Example 2.3.3. Determine if the following sequence converges, and if so, find its limit:

$$a_n = \sqrt{n^2+2021} - \sqrt{n^2+1101n}.$$

Note that:

$$\lim_{n\to\infty} \sqrt{n^2+2021} - \sqrt{n^2+1101n} \cdot \frac{\sqrt{n^2+2021} + \sqrt{n^2+1101n}}{\sqrt{n^2+2021} + \sqrt{n^2+1101n}}.$$

Evaluates to:

$$\lim_{n\to\infty} \frac{(n^2+2021) - (n^2+1101n)}{\sqrt{n^2+2021} + \sqrt{n^2+1101n}}.$$

And the n^2 term cancels out to give:

$$\lim_{n\to\infty} \frac{2021 - 1101n}{\sqrt{n^2+2021} + \sqrt{n^2+1101n}}.$$

From here we can use the technique in section 2.3.1:

$$\lim_{n\to\infty} \frac{2021 - 1101n}{\sqrt{n^2+2021} + \sqrt{n^2+1101n}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{2021}{n} - 1101}{\sqrt{1+\frac{2021}{n^2}} + \sqrt{1+\frac{1101}{n}}}.$$

And therefore:

$$\lim_{n\to\infty} a_n = -\frac{1101}{2}.$$

2.4 Squeeze Theorem

Theorem 9. Squeeze Theorem. Given three sequences $\{a_n\}, \{b_n\}, \{c_n\}$, if there is an integer N_0 such that $\forall n \geqslant N_0$:

$$a_n \leqslant b_n \leqslant c_n.$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

then:

$$\lim_{n\to\infty} b_n = L.$$

Example 2.4.1. Does the following limit exist, and if so, what is its limit?

$$\lim_{n\to\infty} \frac{\cos n}{n}.$$

Note. We cannot use the product rule of limits since the rule follows the assumption that the sequences being multiplied together are both convergent, and we know that $\cos n$ is divergent.

For $n \geqslant 1$, we have the following inequality:

$$-1 \leqslant \cos n \leqslant 1.$$

Multiplying throughout by $\frac{1}{n}$ we obtain:

$$-\frac{1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}.$$

Since we know that:

$$\lim_{n\to\infty} \frac{1}{n} = 0 = \lim_{n\to\infty} -\frac{1}{n}.$$

By the squeeze theorem, we can conclude that:

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

Example 2.4.2. Does the following limit exist, and if so, what is its limit?

$$\lim_{n\to\infty} \frac{3^n}{n!}.$$

For $n \geqslant 4$, we have:

$$0 < \frac{3^n}{n!} = \frac{3}{1}\frac{3}{2}\frac{3}{3}\underbrace{\frac{3}{4}\cdots\frac{3}{n-1}}_{<1}\frac{3}{n}.$$

By replacing terms where $n \geqslant 4$ with 1, we see the following inequality:

$$\frac{3}{1}\frac{3}{2}\frac{3}{3}\underbrace{\frac{3}{4}\cdots\frac{3}{n-1}}_{<1}\frac{3}{n} \leqslant \frac{3}{1}\frac{3}{2}\frac{3}{3}\underbrace{1\cdots1}_{=1}\frac{3}{n} = \frac{27}{2n}.$$

Which gives us the following result:

$$0 < \frac{3^n}{n!} \leqslant \frac{27}{2n}.$$

and we know that:

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{27}{2n} = 0.$$

By the squeeze theorem, we can conclude that:

$$\lim_{n\to\infty} \frac{3^n}{n!} = 0.$$

2.5 Fitting with a Function

Theorem 10. Fitting with a Function. Suppose $\{a_n\}$ is a sequence such that there is a function $f : \mathbb{R} \mapsto \mathbb{R}$ and an integer n_0 such that $a_n = f(n)\ \forall n \geqslant n_0$. Then the following hold true:

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L.$$

$$\lim_{x\to\infty} f(x) = \pm\infty \implies \lim_{n\to\infty} a_n = \pm\infty.$$

Note. When a sequence $\{a_n\}$ can be fitted by a function $f(x)$, we can apply L'Hopital's rule to $\lim_{x\to\infty} f(x)$.

Example 2.5.1. Does the following limit exist, and if so, what is its limit?

$$\lim_{n\to\infty} \frac{3n+e^n}{e^n-n}.$$

Let $f(x) = \frac{3x+e^x}{e^x-x}$. Now note that:

$$\lim_{x\to\infty} \frac{3x+e^x}{e^x-x} \xrightarrow{\text{L'Hopital's Rule}} \lim_{x\to\infty} \frac{3+e^x}{e^x-1} \xrightarrow{\text{L'Hopital's Rule}} \lim_{x\to\infty} \frac{e^x}{e^x} = 1.$$

Thus we have:

$$\lim_{n\to\infty} \frac{3n+e^n}{e^n-n} = \lim_{x\to\infty} \frac{3x+e^x}{e^x-x} = 1.$$

Example 2.5.2. Does the following limit exist, and if so, what is its limit?

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

Let $f(x) = \left(1+\frac{1}{x}\right)^x$. Now note that:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}.$$

Now note that:

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \xrightarrow{\text{L'Hopital's Rule}} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = 1.$$

Thus we have:

$$\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$

We conclude that:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e.$$

2.6 Monotone Convergence Theorem

Definition 3. Non-decreasing, non-increasing, and monotonic sequences. A sequence $\{a_n\}$ is said to be **non-decreasing** for $n \geqslant n_0$ if $a_n \leqslant a_{n+1}$ for every $n \geqslant n_0$ for some $n_0 \in \mathbb{Z}$.

A sequence $\{a_n\}$ is said to be **non-increasing** for $n \geqslant n_0$ if $a_n \geqslant a_{n+1}$ for every $n \geqslant n_0$ for some $n_0 \in \mathbb{Z}$.

A sequence $\{a_n\}$ is said to be **monotonic** if it is either non-decreasing or non-increasing.

Recall that all convergent sequences are bounded, but not all bounded sequences are convergent.

Theorem 11. Monotone Convergence Theorem. Every bounded **monotonic** sequence is convergent.

This theorem is used to show the existence of a limit, but it does not give the value of the limit. It is often used to show that a recursive sequence is convergent.
Example 2.6.1. Consider the following sequence:

$$a_n = \frac{n}{n^2+1}.$$

Show that the sequence is decreasing, explain why it is bounded, find out whether it is convergent.
Let $f(x) = \frac{x}{x^2+1}$. Now note that:

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \ \forall x > 1.$$

Therefore we can say that f is decreasing on $[1, \infty)$. Hence the sequence is decreasing. Since the sequence is decreasing, we have:

$$0 < \underbrace{\frac{n}{n^2+1}}_{a_n} < \cdots < a_3 < a_2 < \frac{1}{2}, \ \forall n \geqslant 1.$$

Therefore the sequence is bounded. Since the sequence is bounded and monotonic, it is convergent.
Example 2.6.2. Consider the following sequence:

$$a_0 = 0,\ a_1 = \sqrt{2},\ a_2 = \sqrt{2+\sqrt{2}},\ a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots$$

Determine if the sequence is bounded, and if it is monotonic, and as a consequence if it is convergent.

Proof. We want to show that the sequence is bounded, and we can use induction to show that:

$$0 \leqslant a_n \leqslant 2.$$

For the base case $n = 0$ this assumption holds true. Assuming that this inequality holds for an arbitrary n, we have to check the case for $n+1$. We know that:

$$a_{n+1} = \sqrt{2 + a_n}.$$

And since we assume in the inductive hypothesis that $a_n \leqslant 2$ we can safely say the following:

$$\sqrt{2 + a_n} \leqslant \sqrt{2 + 2}.$$

Which therefore means that $a_{n+1} \leqslant 2$ thus proving it true $\forall n \in \mathbb{Z}$. Making the sequence bounded. □

Now we need to check if the sequence is monotonic. Looking at the sequence, we can see that it seems to be increasing, we just need to prove this assumption.

Proof. We can use induction to check if:

$$a_n < a_{n+1}.$$

For the base case $n = 0$ we can see that it holds true. Assuming that hits inequality holds for some arbitrary n, we have to check the case for $n + 1$. We know that:

$$\begin{aligned} a_{n+2} &= \sqrt{2 + a_{n+1}} \\ &> \sqrt{2 + a_n} \\ \therefore a_{n+2} &> a_{n+1} \end{aligned}$$

Therefore the sequence is strictly increasing and therefore monotonic. □

Since the sequence is bounded and monotonic, we can conclude that the sequence is convergent. To find the limit of the sequence, we can apply a limit to the following recurrence:

$$a_{n+1} = \sqrt{2 + a_n}.$$

and then re-arrange the result:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n\to\infty} a_n}.$$

Which gives us the following:

$$\begin{aligned} L = \sqrt{2 + L} &\iff L^2 - L - 2 = 0 \\ &\iff (L-2)(L+1) = 0 \\ &\iff L = 2 \text{ or } L = -1 \end{aligned}$$

And since $0 \leqslant L \leqslant 2$ we can conclude that:

$$\lim_{n\to\infty} a_n = 2.$$

Lecture No. 3

Series: A Special Sequence

3.1 Series

Definition 4. Partial Sums & Infinite Series. Consider a sequence $\{a_n\}$. A **partial sum** or an n-th **partial sum** is the sum s_n of finite number of ordered terms defined by:

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \sum_{k=1}^{n} a_k \end{aligned}$$

A series or an **infinite series** is an expression of the form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots + .$$

Note that $\{s_n\}$ forms a sequence of partial sums:

$$s_1, s_2, s_3, \ldots, s_n, \ldots .$$

Using the limit symbol, we have:

$$\sum_{n=1}^{\infty} a_n = \lim_{m \to \infty} \left(\sum_{n=1}^{m} a_n \right).$$

which may or may not exist in general.

Definition 5. Convergent Series & Sum. Given a sequence $\{a_n\}$, if the sequence of partial sums $\{s_n\}$ converges to a number s, then the series $\sum_{n=1}^{\infty} a_n$is said to be **convergent** and its sum is s:

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series.

3.2 Geometric Series

Definition 6. Geometric Series. A **geometric sequence** is a sequence $\{ar^n\}_{n=0}^{\infty}$, where $a \neq 0$ is a constant and $\frac{a_{n+1}}{a_n} = r$ is a constant, known as the common ratio r:

$$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots .$$

The corresponding series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

is called a **geometric series**.

Example 3.2.1. The series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n .$$

is a geometric series with $a = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$.
Example 3.2.2. The series:

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots = \sum_{n=0}^{\infty} (-1)^n .$$

is a geometric series with $a = 1$ and the common ratio $r = -1$.

Theorem 12. The geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots .$$

where $a \neq 0$ is convergent if $|r| < 1$. Its limit is:

$$\frac{a}{1-r} .$$

The geometric series is divergent if $|r| \geqslant 1$.

Proof. Equation for limit of a convergent series. If $r = 1$, the series is of the form:

$$a + a + a + \cdots + a + \cdots .$$

which diverges to ∞. Whereas if $r = -1$, the series is of the form:

$$a - a + a - \cdots + a - \cdots .$$

which never converges to a number. For $|r| \neq 1$, we let:

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n . \tag{3.1}$$

When multiplied by r gives:

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}. \tag{3.2}$$

And if we take equation (3.2) away from equation (3.1), we get the following:

$$(1-r)s_n = a - ar^{n+1}.$$

Which therefore gives us:

$$s_n = \frac{a(1-r^{n+1})}{1-r}.$$

If $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$. It follows that:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r}.$$

Therefore:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| > 1$, then the geometric series diverges since $\lim_{n\to\infty} r^{n+1}$ does not exist.

$\square$

3.3 Telescoping Series

A **telescoping series** is a series which can be expressed as:

$$\sum_{n=1}^{\infty} a_n - a_{n+m},$$

for some m.

Example 3.3.1. Evaluate a_n and m in the following telescoping series:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2+9} - \frac{1}{(n+3)^2+9} \right).$$

We see that $a_n = \frac{1}{n^2+9}$, $m = 3$.

Example 3.3.2. Evaluate a_n and m in the following telescoping series:

$$\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right) \right).$$

We see that $a_n = \cos\left(\frac{\pi}{n}\right)$, $m = 2$.

Consider a telescoping series:

$$\sum_{n=1}^{\infty} a_n - a_{n+m},$$

where m is a fixed integer. Let s_n be the partial sum which is evidently of the following form:

$$s_n = \sum_{n=1}^{N} (a_n - a_{n+m}).$$

For $N > m$, we have:

$$\begin{aligned}
s_n &= \sum_{n=1}^{N}(a_n - a_{n+m}) \\
&= \left(\sum_{n=1}^{N} a_n\right) - \left(\sum_{n=1}^{N} a_{n+m}\right) \\
&= \left(\sum_{n=1}^{m} a_n + \sum_{n=m+1}^{N} a_n\right) - \left(\sum_{n=m+1}^{N} a_n + \sum_{n=N+1}^{N+m} a_n\right) \\
&= \left(\sum_{n=1}^{m} a_n\right) - \left(\sum_{n=N+1}^{N+m} a_n\right)
\end{aligned}$$

which is simply the sum of the first m terms minus the sum of the last m terms. Note that as $N \to \infty$, we have:

$$\sum_{n=1}^{\infty}(a_n - a_{n+m}) = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N}(a_n - a_{n+m})}_{s_n} = \lim_{N\to\infty} s_n.$$

Therefore if $\lim_{N\to\infty} a_n = L$ exists, then:

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - a_{n+m}) &= \lim_{N\to\infty}\left(\sum_{n=1}^{m} a_n - \sum_{n=N=1}^{N+m} a_n\right) \\
&= \sum_{n=1}^{m} a_n - \lim_{N\to\infty} \sum_{n=N+1}^{N+m} a_n \\
&= \sum_{n=1}^{m} a_n - \lim_{N\to\infty} (a_{N+1} + a_{N+2} + \cdots + a_{N+m}) \\
&= \left(\sum_{n=1}^{m} a_n\right) - mL.
\end{aligned}$$

If $\lim_{N\to\infty} a_N$ does not exist, then the telescoping series is divergent.
Example 3.3.3. Consider the telescoping series:

$$\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right)\right).$$

Note that $a_n = \cos\left(\frac{\pi}{n}\right)$, and $m = 2$. For $N > 2$, we have:

$$s_n = \sum_{n=1}^{2}\cos\left(\frac{\pi}{n}\right) - \sum_{n=N+1}^{N+2}\cos\left(\frac{\pi}{n}\right).$$

Note. $\lim_{n\to\infty}\cos\left(\frac{\pi}{n}\right) = \cos 0 = 1$.

Therefore we have:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+2}\right)\right) &= \sum_{n=1}^{2}\cos\left(\frac{\pi}{n}\right)-\lim_{N\to\infty}\sum_{n=N+1}^{N+2}\cos\left(\frac{\pi}{n}\right)\\
&= \left(\cos(\pi)+\cos\left(\frac{\pi}{2}\right)\right)-2(1)\\
&= -1-2\\
&= -3
\end{aligned}$$

The given telescoping series converges to -3.

3.4 Harmonic Series

The **harmonic series** is the following series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}.$$

Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$.

Proof. Divergence of the Harmonic Series.
Note. The sequence of partial sums $\{s_n\}$ has a subsequence of the form $\{s_{2^k}\}$. Taking a look at the subsequence:

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1+\frac{1}{2}\\
s_4 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)\geqslant 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)=1+2\left(\frac{1}{2}\right)\\
s_8 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)\\
&\geqslant 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=1+3\left(\frac{1}{2}\right)
\end{aligned}$$

Principally, we wish to establish the following inequality:

$$s_{2^n}\geqslant 1+n\left(\frac{1}{2}\right).$$

We have verified that the inequality holds true for $n=0,1,2$. Now, suppose the following statement is true:

$$s_{2^n}\geqslant 1+n\left(\frac{1}{2}\right),\ n\in\mathbb{Z}.$$

Now, we consider the case $n+1$:

$$\begin{aligned}
s_{2^{n+1}} &= \underbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}}_{s_{2^n}}+\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}}\\
&\geqslant s_{2^n}+\underbrace{\left(\frac{1}{2^n+2^n}+\frac{1}{2^n+2^n}+\cdots+\frac{1}{2^n+2^n}\right)}_{\text{contains } 2^n \text{ terms}}
\end{aligned}$$

Since $\frac{2^n}{2^{n+1}} = \frac{1}{2}$:

$$\geqslant 1 + n\left(\frac{1}{2}\right) + \frac{1}{2}$$
$$\geqslant 1 + (n+1)\frac{1}{2}$$

Continuing on, we see that:

$$\lim_{n\to\infty} s_{2^n} = \infty.$$

Therefore:

$$\lim_{n\to\infty} s_n = \infty.$$

Making the series divergent. □

3.5 Properties

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are $c\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n - b_n)$, where c is any constant. The following laws apply:

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.
- $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
- $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Because of the properties for convergent series, we have the following:

- Every non-zero constant multiple of a divergent series is divergent.
- If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n \pm b_n)$ diverges.

3.6 Test for Divergence

Theorem 13. Necessary condition for series convergence. If the series:

$$\sum_{n=1}^{\infty} a_n.$$

is a convergent series, then:

$$\lim_{n\to\infty} a_n = 0.$$

Note. $\lim_{n\to\infty} a_n = 0$ does not imply that the series converges (Harmonic Series).

Proof. Note that:

$$a_n = s_n - s_{n-1}.$$

Suppose that:

$$\sum_{n=1}^{\infty} a_n = s.$$

for some number s. Then:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = s.$$

Therefore, we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

□

The above result is useful in its contrapositive form:

If $\lim_{n\to\infty} a_n$ does not exist or does not equal to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Lecture No. 4

Indefinite & Definite Integrals

4.1 Antiderivatives & Indefinite Integrals

Definition 7. Antiderivative. A function F is said to be an **antiderivative** of f on an open interval I if $F'(x) = f(x)\ \forall x \in I$.

Note. An antiderivative of f is defined for f on an open set, which is a union of open intervals.

Theorem 14. Let $F(x)$ and $G(x)$ be two antiderivatives of f on an open interval I. Then, $G(x) = F(x) + C$ on I, for some constant C.

Proof. Let us choose two arbitrary real numbers $x_1, x_2 \in I$ where $x_1 < x_2$. Next, let us construct a function $h(x)$ on $[x_1, x_2]$ and let:

$$h(x) = G(x) - F(x).$$

Since $G(x)$ and $F(x)$ are **anti derivatives** of $f(x)$ on I, the following equations are true:

$$G'(x) = f(x)$$
$$F'(x) = f(x)$$

Both $G(x)$ and $F(x)$ are differentiable on I, which implies that both $G(x)$ and $F(x)$ are continuous on I and in particular on $[x_1, x_2]$. Therefore, we can say that $h(x) = G(x) - F(x)$ is also continuous on $[x_1, x_2]$ and that it is also differentiable on I. By the Mean Value Theorem, we have the following equation that holds true:

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c),\ c \in (x_1, x_2).$$

And now we have:

$$\begin{aligned} h'(c) &= G'(c) - F'(c) \\ &= f(c) - f(c) = 0 \end{aligned}$$

Therefore we have:

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = 0.$$

Which means that:

$$h(x_2) - h(x_1) = 0.$$

Since $x_1, x_2 \in I$, we have:

$$h(x_1) = h(x_2) = C.$$

where C is some constant. We therefore conclude that:

$$\begin{aligned} h(x) &= C \ \forall x \in I \\ G(x) - F(x) &= C \ \forall x \in I \\ \therefore G(x) &= F(x) + C \ \forall x \in I \end{aligned}$$

□

Definition 8. Indefinite Integral. The **indefinite integral** of f, denoted by:

$$\int f(x)\,\mathrm{d}x.$$

is the most general antiderivative of f. The function f is called the **integrand**.

Example 4.1.1. For $I = \mathbb{R}$, we have:

$$\int \cos(x)\,\mathrm{d}x = \sin(x) + C.$$

Where C is an arbitrary constant.
Example 4.1.2. For $I = (0, \infty)$, we have:

$$\int 3x^2 - \frac{2}{\sqrt{x}}\,\mathrm{d}x = x^3 - 4\sqrt{x} + 179 + C.$$

Where C is an arbitrary constant.
By definition, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,\mathrm{d}x\right) = f(x).$$

Thus, to verify $\int f(x)\,\mathrm{d}x = F(x)$, we verify:

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = f(x).$$

Example 4.1.3. Prove that for $I \subseteq (0, \infty)$ or $I \subseteq (-\infty, 0)$, we have:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C.$$

Solution. First we note that it follows from the definition that:

$$\begin{aligned}\int f(x)\,\mathrm{d}x = \ln|x| + C &\iff \frac{\mathrm{d}}{\mathrm{d}x}(\ln|x| + C) = f(x)\\ &\iff \frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \frac{1}{x}\end{aligned}$$

For $I \subseteq (0,\infty)$, we have $x > 0$ and:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

For $I \subseteq (-\infty, 0)$, we have $x < 0$ and:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln -x) = \frac{1}{-x}\cdot -1 = \frac{1}{x}.$$

Therefore we have proven that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \frac{1}{x}.$$

And by definition, we have:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C.$$

4.2 Rules for Integration

Theorem 15. Rules for Integration.

$$\begin{aligned}\int (f(x) + g(x))\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x\\ \int (f(x) - g(x))\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x\\ \int cf(x)\,\mathrm{d}x &= c\int f(x)\,\mathrm{d}x, \text{ where } c \text{ is a constant}\end{aligned}$$

To prove the above properties, we verify that the derivatives of functions on both sides are equal.

Proof. The integral of a sum is the sum of the integrals. We can prove this by differentiating the expression on the right.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,\mathrm{d}x\right) + \left(\int g(x)\,\mathrm{d}x\right)\\ &= f(x) + g(x)\end{aligned}$$

Thus, we have:

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

The same proof method is used for the other rules. □

Example 4.2.1. If $f'(x) = 2x - 3$ and $f(2) = 3$, find $f(x)$ where $x \in \mathbb{R}$.
Solution. Since $f'(x) = 2x - 3$, we have:

$$f(x) = \int 2x - 3\,\mathrm{d}x = x^2 - 3x + C.$$

for some constant C. Given that $f(2) = 3$, we obtain $C = 5$. Thus:

$$f(x) = x^2 - 3x + 5.$$

4.3 The Definite Integral & Area Under a Curve

To find the area under a curve $y = f(x)$, where $f(x) > 0$ from $x = a$ to $x = b$, we divide the interval $[a, b]$ into n equal subintervals:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{k-1}, x_k], \ldots, [x_{n-1}, x_n].$$

The width of each subinterval is:

$$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}.$$

We have $x_0 = a$ and $x_n = b$. Thus, we have:

$$x_k = x_0 + k\left(\frac{b-a}{n}\right) \qquad k = 0, 1, 2, 3, \ldots, n.$$

In each k-th subinterval $[x_{k-1}, x_k]$, we choose a point x_k^* and evaluate the value $f(x_k^*)$. The area of the k-th rectangle, over $[x_{k-1}, x_k]$, with height $f(x_k^*)$, is:

$$f(x_k^*)\Delta x = \frac{b-a}{n} f(x_k^*).$$

Now, we approximate the area under the curve $y = f(x)$ by the total areas of all these rectangles:

$$\sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f(x_k^*).$$

This sum is called a Riemann Sum of f on $[a, b]$. If the function is well-behaved, as we increase the number n of subintervals, the length of subinterval Δx tends to zero, making the approximation reach the area A under the curve:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f(x_k^*).$$

In general, the limit may not exist, however, if it exists ,we say that f is **Riemann** integrable on $[a, b]$. The definite integral of f from a to b, denoted by:

$$\int_a^b f(x)\,\mathrm{d}x.$$

Which is the value of the limit. This value is independent of our choice of x_k^*.

Theorem 16. Existence of Definite Integral. If f is continuous or monotonic or piecewise continuous with a finite number of discontinuities on $[a, b]$, then the definite integral:

$$\int_a^b f(x)\,\mathrm{d}x$$

does exist.

Note that if $a > b$ we define:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

And if $a = b$, we define:

$$\int_a^b f(x)\,\mathrm{d}x = 0.$$

For a general function f, the definite integral:

$$\int_a^b f(x)\,\mathrm{d}x.$$

is the net area between the graph of $y = f(x)$ and the x-axis.
Note. The definite integral is a number which is independent of the variable x:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(s)\,\mathrm{d}s.$$

Where the variables x, t, s are dummy variables.
Note. If f is an odd continuous function. Then:

$$\int_{-a}^a f(x)\,\mathrm{d}x = 0.$$

4.4 Riemann Sum using Equal-width Partition

Consider a continuous function f on $[a, b]$, where the definite integral exists. We shall consider a special Riemann sum and use it to compute the value of the definite integral. Suppose the width of each subinterval is the same:

$$\Delta x = \frac{b-a}{n}.$$

Then:

$$x_k = a + k\left(\frac{b-a}{n}\right) \quad k = 0, 1, 2, \ldots, n.$$

With $x_k^* \in [x_{k-1}, x_k]$, the corresponding Riemann sum of f on $[a, b]$ is:

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*).$$

And the definite integral of f from a to b is:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*).$$

Where the limit of the Riemann sum as $n \to \infty$ must be independent of how the sample points x_k^* are chosen. With the **right endpoints** as sample points, we have the right Riemann sum:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n}\end{aligned}$$

With the **left endpoints** as sample points, we have the left Riemann sum:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{(k-1)(b-a)}{n}\right)\cdot\frac{b-a}{n}\end{aligned}$$

Example 4.4.1. Use the right Riemann sums to find the value of the following definite integral:

$$\int_1^3 x^2\,\mathrm{d}x.$$

Solution. Partition the interval $[1, 3]$ into n subintervals of equal width:

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

Thus we have:

$$x_k = 1 + k\Delta x = 1 + \frac{2k}{n}.$$

And our Riemann sum is of the following form:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n} f\left(1+\frac{2k}{n}\right)\Delta x\\ &= \sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)^2\cdot\frac{2}{n}\\ &= \frac{2}{n}\sum_{k=1}^{n}\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)\\ &= \frac{2}{n}\left(\sum_{k=1}^{n}1+\sum_{k=1}^{n}\frac{4k}{n}+\sum_{k=1}^{n}\frac{4k^2}{n^2}\right)\\ &= \frac{2}{n}\left(n+\frac{4}{n}\sum_{k=1}^{n}k+\frac{4}{n^2}\sum_{k=1}^{n}k^2\right)\\ &= \frac{2}{n}\left(n+\frac{1}{n}\cdot\frac{n(n+1)}{2}+\frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right)\\ &= 2\left(1+2+\frac{2}{n}+\frac{2}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right)\end{aligned}$$

Therefore the definite integral:

$$\int_1^3 x^2\,\mathrm{d}x = \lim_{n\to\infty}\left[2+\left(1+2+\frac{2}{n}+\frac{2}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right)\right] = \frac{26}{3}.$$

4.5 Properties of Definite Integrals

Theorem 17. If the integral on the left hand side of the equation exists, the result on the right is true:

$$\int_a^b c\,\mathrm{d}x = c(b-a)$$
$$\int_a^b Kf(x)\,\mathrm{d}x = K\int_a^b f(x)\,\mathrm{d}x$$
$$\int_a^b (f(x)\pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$$
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

Where K is a constant.

Example 4.5.1. Suppose f is continuous on $\mathbb{R}$, and:

$$\int_0^3 f(x)\,\mathrm{d}x = 3, \qquad \int_0^4 f(x)\,\mathrm{d}x = 7.$$

Find:

$$\int_4^3 f(x)\,\mathrm{d}x.$$

Solution.

$$\begin{aligned}\int_4^3 f(x)\,\mathrm{d}x &= -\int_3^4 f(x)\,\mathrm{d}x\\ &= -\left(\int_0^4 f(x)\,\mathrm{d}x - \int_0^3 f(x)\,\mathrm{d}x\right)\\ &= -(7-3)\\ &= -4\end{aligned}$$

Theorem 18. Suppose the following integrals exist and $a < b$.

$$f(x) \geqslant 0\ \forall x \in [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geqslant 0.$$

$$f(x) \geqslant g(x)\ \forall x \in [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geqslant \int_a^b g(x)\,\mathrm{d}x.$$

$$m \geqslant f(x) \geqslant M\ \forall x \in [a,b] \implies m(b-a) \geqslant \int_a^b f(x)\,\mathrm{d}x \geqslant M(b-a).$$

Lecture No. 5

Fundamental Theorem of Calculus

5.1 The First Fundamental Theorem of Calculus

The fundamental theorem gives the precise inverse relationship between the derivative and the definite integral.

Definition 9. Mean Value. If f is continuous on $[a, b]$, then the **mean value** (also known as the **average value**) of f on $[a, b]$ is:

$$\frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

Example 5.1.1. What is the mean value of $f(x) = x^2$ on the interval $[1, 3]$?
Solution. We had previously calculated in section 4.4 that:

$$\int_1^3 x^2\,\mathrm{d}x = \frac{26}{3}.$$

Thus the mean value of x^2 on $[1, 3]$ is:

$$\begin{aligned} \frac{1}{3-1} \int_1^3 x^2\,\mathrm{d}x &= \frac{1}{3-1} \cdot \frac{26}{3} \\ &= \frac{13}{3}. \end{aligned}$$

Theorem 19. The Mean Value Theorem for Definite Integrals. If f is continuous on $[a, b]$, then $\exists c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

Otherwise written:

$$(b-a)f(c) = \int_a^b f(x)\,\mathrm{d}x.$$

Proof. Mean Value Theorem for Definite Integrals. Suppose f is continuous on the interval $[a, b]$. Then we know that f has a maximum $M = f(s)$ and a minimum $m = f(r)$ on $[a, b]$ by the **Extreme Value Theorem**. By one of the order properties of the definite integral, we then have:

$$m(b-a) \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant M(b-a).$$

And if we divide by $b - a$ we get:

$$f(r) = m \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant M = f(s).$$

By the **Intermediate Value Theorem**, f must take on every value between $f(r)$ and $f(s)$. Therefore, $\exists c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

□

Theorem 20. First Fundamental Theorem of Calculus. Suppose f is continuous on $[a, b]$. Let $F(x)$ be the function defined by:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad a \leqslant x \leqslant b.$$

Then $F(x)$ is continuous on $[a, b]$, and $F(x)$ is differentiable on (a, b) where:

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x).$$

Note. In the integral above, the lower limit a is a constant, and the upper limit of the integral is the variable x.

Proof. First Fundamental Theorem of Calculus. The definite integral:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Is the area under $f(t)$ on the interval $[a, x]$. Note the $F(x)$ is a continuous function for $x \in [a, b]$. Then:

$$\frac{\Delta F}{\Delta x} = \frac{F(x+h) - F(x)}{h} = \frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h} = \frac{h \cdot f(x^*)}{h} = f(x^*).$$

Where x^* is between x and $x + h$. As $h \to 0$, we have $x^* \to x$, and hence:

$$\frac{F(x+h) - F(x)}{h} \to f(x).$$

Thus, we have:

$$F'(x) = f(x).$$

□

Example 5.1.2. Let $b > 1$. Find $g'(x)$, where:

$$g(x) = \int_1^x \frac{\sin(t)}{t}\,\mathrm{d}t, \quad 1 \leqslant x \leqslant b.$$

Solution. By the First Fundamental Theorem of Calculus, the function:

$$g(x) = \int_1^x \frac{\sin(t)}{t}\,\mathrm{d}t$$

is continuous on $[1, b]$ and is differentiable in $(1, b)$ and:

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \frac{\sin(t)}{t}\,\mathrm{d}t\right) = \frac{\sin(x)}{x}.$$

Example 5.1.3. Find the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^\pi e^{(t-3)^2}\,\mathrm{d}t\right).$$

Note. The lower limit of the integral is not a constant.
Solution.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^\pi e^{(t-3)^2}\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_\pi^x e^{(t-3)^2}\,\mathrm{d}t\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_\pi^x e^{(t-3)^2}\,\mathrm{d}t\right)\\ &= -e^{(x-3)^2}.\end{aligned}$$

Note. In general, we have the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{u(x)} f(t)\,\mathrm{d}t = u'(x)\cdot f(u(x)).$$

Proof. Let $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$. Then:

$$\int_a^{u(x)} f(t)\,\mathrm{d}t = F(u(x)) = (F \circ u)(x).$$

Therefore, we apply the **Chain Rule** to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_a^u f(t)\,\mathrm{d}t\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = u'(x)\cdot f(u(x)).$$

□

Example 5.1.4. Find the derivative of:

$$F(x) = \int_{x^2}^{x^3} e^{-t^2}\,\mathrm{d}t.$$

Solution. First note that:

$$F(x) = \int_{x^2}^{x^3} e^{-t^2}\,\mathrm{d}t = \int_0^{x^3} e^{-t^2}\,\mathrm{d}t - \int_0^{x^2} e^{-t^2}\,\mathrm{d}t.$$

Thus we have:

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x^3} e^{-t^2}\,\mathrm{d}t\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x^2} e^{-t^2}\,\mathrm{d}t\right) \\ &= 3x^2 e^{-x^6} - 2xe^{-x^4}. \end{aligned}$$

5.2 The Second Fundamental Theorem of Calculus

Theorem 21. The Second Fundamental Theorem of Calculus. If f is continuous on $[a, b]$, then:

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a),$$

where G is any continuous antiderivative of f on $[a, b]$ such that:

$$G' = f.$$

We write:

$$\int_a^b f(x)\,\mathrm{d}x = G(x)\Big|_a^b.$$

Proof. Second Fundamental Theorem of Calculus. We know that:

$$G'(x) = f(x).$$

And by the first fundamental theorem of calculus:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x).$$

Hence, both $G(x)$ and $\displaystyle\int_a^x f(t)\,\mathrm{d}t$ are antiderivatives of $f(x)$, and they must differ by a constant on (a, b):

$$G(x) = \int_a^x f(t)\,\mathrm{d}t + C, \quad \forall x \in (a, b),$$

for some constant C. Next, we have:

$$\begin{aligned} G(b) - G(a) &= \left(\int_a^b f(t)\,\mathrm{d}t + C\right) - \left(\int_a^a f(t)\,\mathrm{d}t + C\right) \\ &= \int_a^b f(t)\,\mathrm{d}t. \end{aligned}$$

□

Example 5.2.1. Evaluate the following:

$$\int_1^4 x^{-2} + 3\sqrt{x} - \frac{1}{\sqrt{x}}\,\mathrm{d}x.$$

Solution.

$$\begin{aligned}\int_1^4 x^{-2} + 3\sqrt{x} - \frac{1}{\sqrt{x}}\,\mathrm{d}x &= -x^{-1} + 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \left(\frac{\sqrt{x}}{\frac{1}{2}}\right)\Bigg|_1^4 \\ &= \left(-\frac{1}{4}+1\right) + 2(8-1) - 2(2-1) \\ &= 12\frac{3}{4}.\end{aligned}$$

5.3 Application

Example 5.3.1. Evaluate:

$$\lim_{n\to\infty}\frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right).$$

Solution. Express the limit as a definite integral:

$$\int_a^b g(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}g(x_k^*),$$

with $x_k^* \in \left[a + \frac{(k-1)(b-a)}{n}, a + \frac{k(b-a)}{n}\right]$. We get:

$$\int_0^1 g(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}g(x_k^*),$$

with $x_k^* = \frac{k}{n}$. Therefore we rewrite the sum in the question:

$$\begin{aligned}\frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \frac{n\pi}{n}\right) &= \sum_{k=1}^{n}\frac{\pi}{n}\sin\frac{k\pi}{n} \\ &= \sum_{k=1}^{n}\frac{1}{n}\pi\sin\left(\left(\frac{k}{n}\right)\pi\right).\end{aligned}$$

Equating the expression above with our Riemann sum:

$$\sum_{k=1}^{n}\frac{1}{n}g\left(\frac{k}{n}\right) \equiv \sum_{k=1}^{n}\frac{1}{n}\sin\left(\left(\frac{k}{n}\right)\pi\right).$$

By replacing $\frac{k}{n}$ by x, we take $g(x) = \pi\sin(\pi x)$ over $[0,1]$. Thus we have:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right) = \int_0^1 \pi\sin(\pi x)\,\mathrm{d}x = \pi\left(-\frac{1}{\pi}\cos(\pi x)\right)\Bigg|_0^1 = 2.$$

As required.

Lecture No. 6

Substitution & Integration by Parts

6.1 The Substitution Rule

The **Substitution Rule** is a consequence of the **Chain Rule**:

$$\int f(u(x))\,\underbrace{u'(x)\,\mathrm{d}x}_{\mathrm{d}u} = \int f(u)\,\mathrm{d}u.$$

The idea behind the **substitution rule** is to replace a relatively complicated integral by a simpler integral.

Example 6.1.1. Evaluate the following integral:

$$\int x^4 \sin\left(x^5\right)\,\mathrm{d}x.$$

Solution. Choose u to be some integrand whose derivative also occurs:

$$\int x^4 \sin\left(x^5\right)\,\mathrm{d}x = \frac{1}{5}\int \sin\left(x^5\right)\cdot 5x^4\,\mathrm{d}x$$

Substitute $u(x) = x^5$, $u'(x) = 5x^4$:

$$\begin{aligned}
&= \frac{1}{5}\int \underbrace{\sin\left(x^5\right)}_{\sin(u)}\cdot \underbrace{5x^4\,\mathrm{d}x}_{u'(x)\,\mathrm{d}u} \\
&= \frac{1}{5}\int \sin(u)\,\mathrm{d}u \\
&= \frac{1}{5}((-\cos(u)) + C) \\
&= -\frac{1}{5}\cos(x^5) + C.
\end{aligned}$$

Example 6.1.2. Evaluate the following integral:

$$\int \frac{x}{x^2+1}\,\mathrm{d}x.$$

Solution. Note that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)=2x.$$

Let $u=x^2+1$, $u'=2x$:

$$\begin{aligned}\int \frac{x}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{x^2+1}(2x)\,\mathrm{d}x\\ &= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u\\ &= \frac{1}{2}\ln|u|+C\\ &= \frac{1}{2}\ln\left(x^2+1\right)+C\\ &= \ln\sqrt{x^2+1}+C.\end{aligned}$$

When using the **substitution rule** for definite integrals, change the upper and lower limits of the definite integral together with the substitution u:

$$\int_a^b f(u(x))u'(x)\,\mathrm{d}x=\int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u.$$

Example 6.1.3. Evaluate the following definite integral:

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,\mathrm{d}x.$$

Solution. Choose $u=\sqrt{x+1}$, $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x+1}}$, then:

$$x=0 \implies u=1, \qquad x=8 \implies u=3.$$

Thus we have:

$$\begin{aligned}\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,\mathrm{d}x &= \int_0^8 2\left(\cos\sqrt{x+1}\right)\frac{1}{2\sqrt{x+1}}\,\mathrm{d}x\\ &= \int_1^3 2\cos u\,\mathrm{d}u\\ &= 2\sin u\Big|_1^3\\ &= 2(\sin(3)-\sin(1))\end{aligned}$$

6.2 Integration by Parts

Integration by Parts is a consequence of the **Product Rule** for differentiation:

$$\int u(x)\underbrace{v'(x)\,\mathrm{d}x}_{\mathrm{d}v}=u(x)v(x)-\int v(x)\underbrace{u'(x)\,\mathrm{d}x}_{\mathrm{d}u}.$$

In short:

$$\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u.$$

Example 6.2.1. Evaluate the following integral:

$$\int x^2 \ln x \, \mathrm{d}x.$$

Solution. Let $u = \ln x$, $v' = x^2$, then:

$$u' = \frac{1}{x}, \qquad v = \frac{x^3}{3}.$$

Thus we have:

$$\begin{aligned}\int x^2 \ln x \, \mathrm{d}x &= \left(\frac{x^3}{3} \ln x\right) - \int \frac{1}{x} \cdot \frac{x^3}{3} \, \mathrm{d}x \\ &= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, \mathrm{d}x \\ &= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.\end{aligned}$$

Example 6.2.2. Evaluate the following integral:

$$\int (t+1)e^t \, \mathrm{d}t.$$

Solution. Let $u = t + 1$, $v'e^t$, then:

$$u' = 1, \qquad v = e^t.$$

Thus we have:

$$\begin{aligned}\int (t+1)e^t \, \mathrm{d}t &= (t+1)e^t - \int e^t \, \mathrm{d}t \\ &= (t+1)e^t - e^t + C \\ &= te^t + C.\end{aligned}$$

Example 6.2.3. Evaluate the following integral:

$$\int \tan^{-1} x \, \mathrm{d}x.$$

Solution. Let $u = \tan^{-1} x$, $v' = 1$, then:

$$u' = \frac{1}{1+x^2}, \qquad v = x.$$

Thus we have:

$$\int \tan^{-1} x \, \mathrm{d}x = x \tan^{-1} x - \int \frac{x}{1+x^2} \, \mathrm{d}x$$

We evaluate the integral by substituting $w = 1 + x^2$:

$$\begin{aligned}&= x \tan^{-1} x - \int \frac{1}{2} \cdot \frac{1}{u} \, \mathrm{d}u \\ &= x \tan^{-1} x - \frac{1}{2} \ln |u| + C \\ &= x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.\end{aligned}$$

6.2.1 The LIPET Rule

When doing integration by parts, certain choices of u work better in general. This is where the **LIPET** rule comes in. It is an acronym, where each letter represents a different type of function:

$$\begin{aligned}
\mathbf{L} &= \text{Logarithmic functions}\\
\mathbf{I} &= \text{Inverse trigonometric functions}\\
\mathbf{P} &= \text{Polynomial functions}\\
\mathbf{E} &= \text{Exponential functions}\\
\mathbf{T} &= \text{Trigonometric functions}
\end{aligned}$$

This gives a systematic list of functions to try and set equal to u in the integration by parts formula.

6.3 Reduction Formulas

Consider the integral:

$$I_n = \int x^n e^x \, \mathrm{d}x.$$

Where n is a non-negative integer. Then for $n \geqslant 1$, the formula:

$$I_n = x^n e^x - nI_{n-1},$$

expresses I_n in terms of I_{n-1}. This is known as a **reduction formula** for:

$$I_n = \int x^n e^x \, \mathrm{d}x.$$

This uses integration by parts recursively to integrate expressions.

Proof. For $n \geqslant 1$, use integration by parts, with:

$$u(x) = x^n, \qquad v'(x) = e^x,$$

so that:

$$u'(x) = nx^{n-1}, \qquad v(x) = e^x.$$

This gives us:

$$\begin{aligned}
I_n &= \int x^n e^x \, \mathrm{d}x\\
&= x^n e^x - \int n(x^{n-1}) e^x \, \mathrm{d}x\\
&= x^n e^x - n \underbrace{\int x^{n-1} e^x \, \mathrm{d}x}_{I_{n-1}}\\
&= x^n e^x - nI_{n-1}.
\end{aligned}$$

□

Example 6.3.1. Let $I_n = \displaystyle\int x^n e^x \, \mathrm{d}x$, where $n \geqslant 0$. Use the reduction formula to determine a formula for I_4.

Solution. We have:

$$\begin{aligned} I_4 &= x^4 e^x - 4I_3 \\ I_3 &= x^3 e^x - 3I_2 \\ I_2 &= x^2 e^x - 2I_1 \\ I_1 &= x e^x - I_0. \end{aligned}$$

Note that $I_0 = \displaystyle\int e^x \, \mathrm{d}x = e^x + C$. Thus, we obtain:

$$I_4 = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + 24C.$$

Lecture No. 7

Trigonometric Functions

7.1 Intergrating $\sin^m x \cos^n x$

To integrate integrands of the following form:

$$\int \sin^m x \cos^n x \, \mathrm{d}x,$$

the following tools are required:

$$\int \cos x \, \mathrm{d}x = \sin x + C \qquad \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\cos^2 x + \sin^2 x = 1$$
$$\int \sin x \, \mathrm{d}x = -\cos x + C \qquad \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

In the case that there is an odd power of $\cos x$, our aim is to substitute $u = \sin x$, with $u' = \cos x$. We take out one $\cos x$ term, and replace the remaining even power of $\cos x$ using:

$$\cos^2 x = 1 - \sin^2 x.$$

And then use our substitution $u = \sin x$. We do the converse when we have and odd power of $\sin x$.

Example 7.1.1. Evaluate the following integral:

$$\int \sin^3 x \cos^8 x \, \mathrm{d}x.$$

Solution.

$$\begin{aligned}
\int \sin^3 x \cos^8 x \, \mathrm{d}x &= \int \sin x (\sin^2 x) \cos^8 x \, \mathrm{d}x \\
&= \int \sin x (1 - \cos^2 x) \cos^8 x \, \mathrm{d}x
\end{aligned}$$

Substituting $u = \cos x$, $u' = -\sin x$:

$$
\begin{aligned}
&= \int -(1-u^2)(u^8)\,\mathrm{d}u \\
&= \int -u^8 + u^{10}\,\mathrm{d}u \\
&= -\frac{u^9}{9} + \frac{u^{11}}{11} + C \\
&= -\frac{\cos^9 x}{9} + \frac{\cos^{11} x}{11} + C.
\end{aligned}
$$

In the case where both powers of $\sin x$ and $\cos x$ are even, we use the double angle formulae to express the integrand as a function of $\cos 2x$.

Example 7.1.2. Evaluate the following integral:

$$\int \sin^4 x \cos^2 x\,\mathrm{d}x.$$

Solution.

$$
\begin{aligned}
\int \sin^4 \cos^2 x\,\mathrm{d}x &= \int (\sin^2 x)^2 \cos^2 x\,\mathrm{d}x \\
&= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) \mathrm{d}x \\
&= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)(1+\cos 2x)\,\mathrm{d}x \\
&= \frac{1}{8}\int 1 - \cos 2x - \underbrace{\cos^2 2x}_{\text{even power}} + \underbrace{\cos^3 2x}_{\text{odd power}}\,\mathrm{d}x \\
&= \frac{1}{8}\int 1 - \cos 2x - \frac{1+\cos 4x}{2} + \cos 2x\left(1 - \sin^2 2x\right)\mathrm{d}x \\
&= \frac{1}{8}\int \frac{1}{2} - \frac{\cos 4x}{2} - \cos 2x \sin^2 2x\,\mathrm{d}x \\
&= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C.
\end{aligned}
$$

7.2 Intergrating $\sec^m x \tan^n x$

To intgrate integrands of the following form:

$$\int \sec^m x \tan^n x\,\mathrm{d}x,$$

we must first obtain some simple derivatives. First, we must find the integral of $\tan x$:

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x.$$

We let $u = \cos x$, therefore $u' = -\sin x$. Then:

$$\begin{aligned}\int \frac{\sin x}{\cos x}\,\mathrm{d}x &= -\int \frac{1}{u}\,\mathrm{d}u \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

Second, we must find the integral of $\sec x$:

$$\begin{aligned}\int \sec x\,\mathrm{d}x &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\,\mathrm{d}x \\ &= \ln|\tan x + \sec x| + C.\end{aligned}$$

Note. We used the following facts to achieve the result above:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x + \sec x] = \sec^2 x + \sec x \tan x, \qquad \int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C.$$

The rest of the following tools are required for this section:

$$\begin{aligned}\int \sec^2 x\,\mathrm{d}x &= \tan x + C \\ \int \sec x \tan x\,\mathrm{d}x &= \sec x + C\end{aligned} \qquad 1 + \tan^2 x = \sec^2 x.$$

In the case where there are odd powers of $\sec x$ **and** $\tan x$, our aim is to substitute $u = \sec x$, with $u' = \sec x \tan x$. First, we keep one copy of $\sec x \tan x$ separate. Then we change the remaining $\tan^n x$ term to its various $\sec x$ terms using:

$$\tan^2 x = \sec^2 x - 1,$$

so that the expression involves $\sec x$ only. Finally, we use our substitution $u = \sec x$.

Example 7.2.1. Evaluate the following integral:

$$\int \sec x \tan^3 x\,\mathrm{d}x.$$

Solution.

$$\begin{aligned}\int \sec x \tan^3 x\,\mathrm{d}x &= \int (\sec x \tan x)\tan^2 x\,\mathrm{d}x \\ &= \int (\sec x \tan x)(\sec^2 x - 1)\,\mathrm{d}x\end{aligned}$$

We use the substitution $u = \sec x$:

$$\begin{aligned}
&= \int u^2 - 1 \, \mathrm{d}u \\
&= \frac{u^3}{3} - u + C \\
&= \frac{\sec^3 x}{3} - \sec x + C.
\end{aligned}$$

In the case where there is a even power of $\sec x$, our aim is to substitute $u = \tan x$, with $u' = \sec^2 x$. First we keep one $\sec^2 x$ term separate. Then, we change the remaining even power $\sec x$ term into powers of $\tan x$ using:

$$\sec^2 x = 1 + \tan^2 x.$$

Finally, we use our substitution $u = \tan x$.

Example 7.2.2. Evaluate the following integral:

$$\int \sec^4 x \tan^3 x \, \mathrm{d}x.$$

Solution.

$$\begin{aligned}
\int \sec^4 x \tan^3 x \, \mathrm{d}x &= \int (\sec^2 x) \sec^2 x \tan^3 x \, \mathrm{d}x \\
&= \int (\sec^2 x)(1 + \tan^2 x) \tan^3 x \, \mathrm{d}x
\end{aligned}$$

We use the substitution $u = \tan x$:

$$\begin{aligned}
&= \int (1 + u^2) u^3 \, \mathrm{d}u \\
&= \int u^3 + u^5 \, \mathrm{d}u \\
&= \frac{u^4}{4} + \frac{u^6}{6} + C \\
&= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C.
\end{aligned}$$

Lecture No. 8

Partial Fractions & Inverse Trigonometric Substitution

8.1 Partial Fractions for Integration of Rational Functions

Now we consider integrals such as:

$$\int \frac{x^3+3x^2}{x^2+1}\,\mathrm{d}x \qquad \int \frac{2+3x+x^2}{x(x^2+1)}\,\mathrm{d}x,$$

or, in general, integrals of the form:

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

where P and Q are polynomials. This is done by **partial fractions**. First we check if the integrand is proper, that is, $\deg(P(x)) < \deg(Q(x))$. If it is not, do long division to divide $P(x)$ by $Q(x)$ until the remainder $R(x)$ has a degree lesser than $Q(x)$:

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \int s(x) + \frac{R(x)}{Q(x)}\,\mathrm{d}x.$$

Then, we factorise the denominator $Q(x)$ as a product of linear and irreducible quadratic factors. We express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of partial fractions of the form:

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}.$$

8.1.1 Linear Factors

If the denominator is a product of distinct linear factors, then the corresponding partial fraction representation for each factor $ax+b$ is $\frac{A}{ax+b}$.

Example 8.1.1. Evaluate $\displaystyle\int \frac{x+2}{x(x-1)(x+1)}\,\mathrm{d}x$.

Solution. There are three distinct linear factors, the corresponding partial fractions are:

$$\frac{A}{x}, \frac{B}{x-1}, \frac{C}{x+1}.$$

Thus, we shall solve for constants A, B, and C such that:

$$\begin{aligned}\frac{x+2}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x+1)(x-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}.\end{aligned}$$

Comparing coefficients in the numerator, we get the following simultaneous equations:

$$\begin{aligned}A + B + C &= 0 \\ B - C &= 1 \\ -A &= 2\end{aligned}$$

The solutions of which are:

$$A = -2, \ B = \frac{3}{2}, \ C = \frac{1}{2}.$$

Thus we have:

$$\begin{aligned}\int \frac{x+2}{x(x-1)(x+1)} \, \mathrm{d}x &= \int -\frac{2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1} \, \mathrm{d}x \\ &= -2\int \frac{1}{x} \, \mathrm{d}x + \frac{3}{2}\int \frac{1}{x-1} \, \mathrm{d}x + \frac{1}{2}\int \frac{1}{x+1} \, \mathrm{d}x \\ &= -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

If $Q(x)$ is a product of linear factors, some of which are **repeated**, say $(ax+b)^k$, where $k \geqslant 2$, then there are k corresponding partial fractions:

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \ldots, \frac{A_{k-1}}{(ax+b)^{k-1}}, \frac{A_k}{(ax+b)^k}.$$

Example 8.1.2. Evaluate $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2} \, \mathrm{d}x$.

Solution. For $(x-3)$, the corresponding partial fraction is $\dfrac{A}{x-3}$. For $(x+2)^2$, the corresponding partial fractions are:

$$\frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

Next we find constants A, B, and C such that:

$$\begin{aligned}\frac{x^2}{(x-3)(x+2)^2} &= \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \\ &= \frac{A(x+2)^2 + B(x-3)(x+2) + C(x-3)}{(x-3)(x+2)^2}.\end{aligned}$$

Comparing coefficients we get:

$$\begin{aligned} A + B &= 1 \\ 4A - B + C &= 0 \\ 4A - 6B - 3C &= 0 \end{aligned}$$

The solutions of which are:

$$A = \frac{9}{25},\ B = \frac{16}{25},\ C = -\frac{4}{5}.$$

Thus we have:

$$\begin{aligned} \int \frac{x^2}{(x-3)(x+2)^2}\,\mathrm{d}x &= \frac{1}{25}\int \frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2}\,\mathrm{d}x \\ &= \frac{9}{25}\int \frac{1}{x-3}\,\mathrm{d}x + \frac{16}{25}\int \frac{1}{x+2}\,\mathrm{d}x - \frac{20}{25}\int \frac{1}{(x+2)^2}\,\mathrm{d}x \\ &= \frac{1}{25}\left(9\ln|x-3| + 16\ln|x+2| + \frac{20}{x+2}\right) + C. \end{aligned}$$

8.1.2 Irreducible Quadratic Factors

The quadratic expression $ax^2 + bx + c$ is said to be irreducible when it cannot be reduced to a product of linear factors. In this case we have $b^2 - 4ac < 0$. We can therefore express the quadratic expression in the form $(Ax + B)^2 + D^2$, via completing the square.

$$f(x) = \frac{P(x)}{Q(x)}.$$

Suppose $Q(x)$ contains the irreducible quadratic factor $ax^2 + bx + c$. Then the partial fraction representation of $f(x)$ will contain the term:

$$\frac{Ax + B}{ax^2 + bx + c}.$$

In this case, we have the useful standard integral:

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

Example 8.1.3. Evaluate $\displaystyle\int \frac{1}{x^2 + 4x + 5x}\,\mathrm{d}x.$

Solution. The quadratic expression $x^2 + 4x + 5$ is irreducible as its discriminant is less than 0. We first complete the square to obtain:

$$x^2 + 4x + 5 = (x+2)^2 + 1.$$

Thus we have:

$$\int \frac{1}{x^2 + 4x + 5}\,\mathrm{d}x = \int \frac{1}{(x+2)^2 + 1}\,\mathrm{d}x.$$

We then use the substitution $u(x) = x + 2$, $\therefore u'(x) = 1$:

$$\begin{aligned}\int \frac{1}{x^2+4x+5}\,\mathrm{d}x &= \int \frac{1}{(x+2)^2+1}\,\mathrm{d}x \\ &= \int \frac{1}{u^2+1}\,\mathrm{d}u \\ &= \tan^{-1} u + C \\ &= \tan^{-1}(x+2) + C.\end{aligned}$$

Example 8.1.4. Evaluate $\displaystyle\int \frac{1}{4x^2+4x+26}\,\mathrm{d}x$.
Solution. Completing the square we have:

$$4x^2 + 4x + 26 = (2x+1)^2 + 5^2.$$

Therefore we have:

$$\int \frac{1}{4x^2+4x+26}\,\mathrm{d}x = \int \frac{1}{(2x+1)^2+5^2}\,\mathrm{d}x$$

Substituting $u(x) = 2x + 1$, $\therefore u'(x) = 2$:

$$\begin{aligned}&= \frac{1}{2}\int \frac{1}{u^2+5^2}\,\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\right) + C \\ &= \frac{1}{10}\tan^{-1}\frac{2x+1}{5} + C.\end{aligned}$$

Another useful standard integral to use in cases where there are irreducible quadratic factors:

$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + C.$$

Example 8.1.5. Evaluate $\displaystyle\int \frac{x}{x^2+4x+13}\,\mathrm{d}x$.
Solution. We can see that $x^2+4x+13$ is irreducible, so we complete the square:

$$x^2 + 4x + 13 = (x+2)^2 + 9.$$

Now we express in terms of partial fractions:

$$\frac{x}{x^2+4x+13} = \underbrace{\frac{A(2x+4)}{x^2+4x+13}}_{\frac{Af'(x)}{f(x)}} + \frac{B}{x^2+4x+13}.$$

Which gives $A = \frac{1}{2}$ and $B = -2$. Therefore we have:

$$\begin{aligned}\int \frac{x}{x^2+4x+13}\,\mathrm{d}x &= \frac{1}{2}\int \frac{2x+4}{x^2+4x+13}\,\mathrm{d}x - 2\int \frac{1}{x^2+4x+13}\,\mathrm{d}x \\ &= \frac{1}{2}\ln|x^2+4x+13| - 2\left(\frac{1}{3}\tan^{-1}\frac{x+2}{3}\right) + C \\ &= \frac{1}{2}\ln|x^2+4x+13| - \frac{2}{3}\tan^{-1}\frac{x+2}{3} + C.\end{aligned}$$

Suppose $Q(x)$ contains the repeating irreducible quadratic factors $(ax^2+bx+c)^k$. Then the partial fraction representation of $f(x)$ will contain the terms:

$$\frac{A_i x + B_i}{(ax^2+bx+c)^i}, \quad \forall i = 1, 2, 3, \ldots, k.$$

Example 8.1.6. Evaluate $\displaystyle\int \frac{x^2}{x(x^2+4)^3}\,\mathrm{d}x$.
Solution. We break the fraction into its partial fractions:

$$\frac{x^2}{x(x^2+4)^3} = \frac{A}{x} + \frac{A_1 x + B_1}{x^2+4} + \frac{A_2 x + B_2}{(x^2+4)^2} + \frac{A_3 x + B_3}{(x^2+4)^3}.$$

Then, we proceed like the previous examples to solve for the unknowns.

8.2 Inverse Trigonometric Substitution

8.2.1 Integrands Involving $\sqrt{a^2+x^2}$ or a^2+x^2

For the inverse substitution, we use $x = a\tan\theta$ instead of the equivalent $\theta = \tan^{-1}\frac{x}{a}$. This is usually useful because:

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

And:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec^2\theta.$$

Example 8.2.1. Evaluate $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x$, where $a > 0$.

Solution. Let $x = a\tan\theta$ so that $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec^2\theta$, and

$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = a\sec\theta.$$

Thus we have:

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x &= \int \frac{1}{a\sec\theta}(a\sec^2\theta)\,\mathrm{d}\theta \\
&= \int \sec\theta\,\mathrm{d}\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.
\end{aligned}$$

Example 8.2.2. Evaluate $\displaystyle\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x$.

Solution. This type of integral is often seen in the last case of integration by partial fractions. Let $x = 2\tan u$. Then $\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$:

$$\begin{aligned}\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x &= \int \frac{1}{(4\sec^2 u)^2}(2\sec^2 u)\,\mathrm{d}u \\ &= \frac{1}{8}\int \frac{1}{\sec^2 u}\,\mathrm{d}u \\ &= \frac{1}{8}\int \cos^2 u\,\mathrm{d}u \\ &= \frac{1}{8}\int \left(\frac{\cos(2u)+1}{2}\right)\mathrm{d}u \\ &= \frac{1}{16}\left(\frac{\sin(2u)}{2}+u\right)+C.\end{aligned}$$

From $u = \tan^{-1}\frac{x}{2}$, we have:

$$\tan u = \frac{x}{2}, \quad \sin u = \frac{x}{\sqrt{4+x^2}}, \quad \cos u = \frac{2}{\sqrt{4+x^2}},$$

so that:

$$\begin{aligned}\frac{\sin(2u)}{2} &= \sin u\cos u \\ &= \frac{2x}{4+x^2}.\end{aligned}$$

Therefore we have:

$$\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x = \frac{1}{16}\left(\frac{2x}{4+x^2}+\tan^{-1}\left(\frac{x}{2}\right)\right)+C.$$

8.2.2 Integrands Involving $\sqrt{a^2-x^2}$ or a^2-x^2

For the inverse substitution, we use $x = a\sin\theta$ instead of the equivalent $\theta = \sin^{-1}\frac{x}{a}$. Thus we have:

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta,$$

and:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta.$$

Example 8.2.3. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4-x}}\,\mathrm{d}x$.

Solution. Let $x = 2\sin\theta$. We have $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$ and:

$$4 - x^2 = 4\cos^2\theta.$$

Thus we have:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x &= \int \frac{1}{4\sin^2\theta(2\cos\theta)}(2\cos\theta)\,\mathrm{d}\theta \\ &= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,\mathrm{d}\theta \\ &= \frac{1}{4}\int \csc^2\theta\,\mathrm{d}\theta \\ &= -\frac{1}{4}\cot\theta + C.\end{aligned}$$

Now, using the identity:

$$1+\cot^2\theta = \csc^2\theta,$$

we have:

$$\cot^2\theta = \csc^2\theta - 1 = \left(\frac{2}{x}\right)^2 - 1 = \frac{4-x^2}{x^2}.$$

Thus, we have:

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x = -\frac{\sqrt{4-x^2}}{4x} + C.$$

8.2.3 Integrands Involving $\sqrt{x^2-a^2}$ or x^2-a^2

For the inverse substitution we use $x-a\sec\theta$ instead of the equivalent $\theta = \sec^{-1}\frac{x}{a}$. Thus we have:

$$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta,$$

and:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec\theta\tan\theta.$$

Example 8.2.4. Find $\dfrac{\sqrt{x^2-a^2}}{x}\,\mathrm{d}x$.

Solution. Let $x = a\sec\theta$. We have $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec\theta\tan\theta$ and:

$$\sqrt{x^2-a^2} = a\tan\theta.$$

Thus we have:

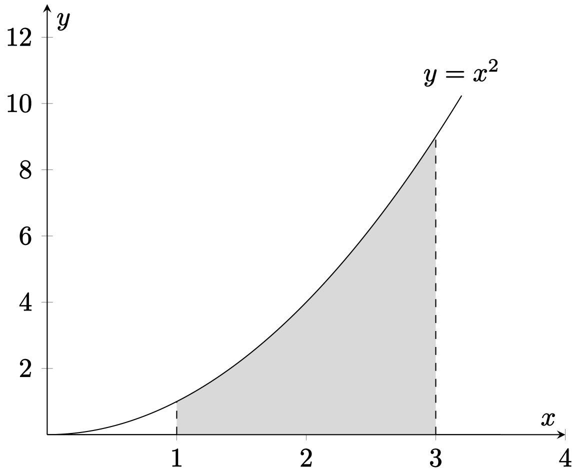
$$\begin{aligned}\int \frac{\sqrt{x^2-a^2}}{x}\,\mathrm{d}x &= \int \frac{a\tan\theta}{a\sec\theta}a\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int a\tan^2\theta\,\mathrm{d}\theta \\ &= a\int(\sec^2\theta) - 1\,\mathrm{d}\theta \\ &= a(\tan\theta - \theta) + C \\ &= \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C.\end{aligned}$$

Lecture No. 9

Applications of Integration

9.1 Area Under a Curve

Example 9.1.1. Find the area of the region enclosed by the curve $y = x^2$, $x = 1$, $x = 3$, and $y = 0$.



Solution. For $1 \leqslant x \leqslant 3$, the area of a typical strip is:

$$x^2 \cdot \delta x.$$

Thus the area of the bounded region is:

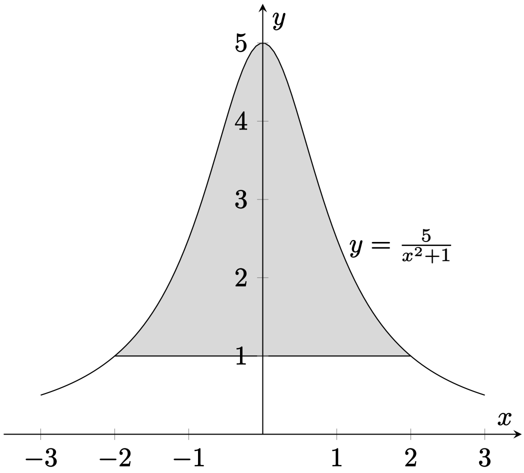
$$\int_1^3 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_1^3 = \frac{26}{3}.$$

Example 9.1.2. Find the area of the region lying above the line $y = 1$ and below the curve $y = \dfrac{5}{x^2+1}$.

Solution. To find the intersection points, we must solve:

$$1 = \frac{5}{x^2+1},$$

which gives $x^2 + 1 = 5$, so $x^2 = 4$ and $x = \pm 2$.



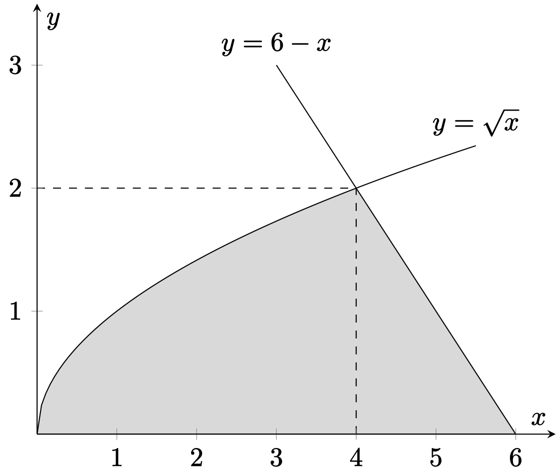
For $-2 \leqslant x \leqslant 2$, the area of a typical strip is:

$$\left(\frac{5}{x^2+1} - 1\right) \cdot \delta x.$$

Therefore the area of the region is then given by:

$$\begin{aligned}\int_{-2}^{2} \frac{5}{x^2+1} - 1 \, \mathrm{d}x &= \left[(5\tan^{-1} x) - x\right]_{-2}^{2} \\ &= 5(\tan^{-1} 2 - \tan^{-1}(-2)) - 4 \\ &= 10\tan^{-1}(2) - 4.\end{aligned}$$

Example 9.1.3. Evaluate the area of the region bounded on the left by $y = \sqrt{x}$, and on the right by $y = 6 - x$.



For $0 \leqslant y \leqslant 2$, note that:

$$y = 6 - x \implies x = 6 - y,$$

and:

$$y = \sqrt{x} \implies x = y^2.$$

The area of a typical horizontal strip is given by:

$$((6-y) - y^2) \cdot \delta y.$$

Therefore, the area of the bounded region is given by:

$$\begin{aligned}\int_0^2 6 - y - y^2 \, \mathrm{d}y &= \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 12 - 2 - \frac{8}{3} \\ &= \frac{22}{3}.\end{aligned}$$

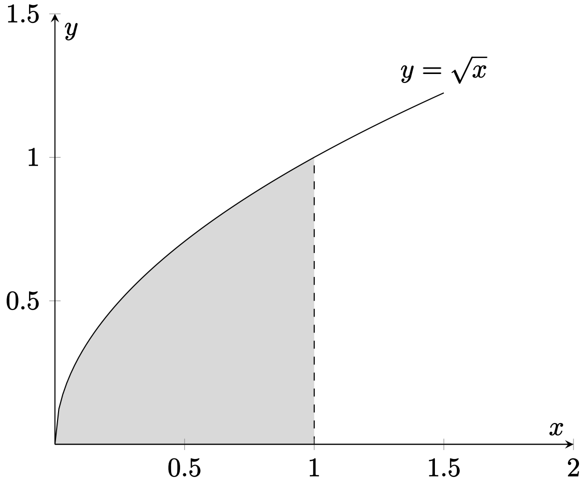
9.2 Volume of Solid of Revolution

9.2.1 The Disc Method

Solids of revolution are solids obtained by revolving a region about a line. The volume of a disc is $\pi r^2 h$ where r is the radius, and h is the height of the disc. To find the volume of a solid created by the revolution of an area around an axis, we substitute r to be the length of a typical horizontal strip, and we substitute h with our infinitesimal:

$$V = \pi \int_a^b (f(x))^2 \, \mathrm{d}x.$$

Example 9.2.1. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



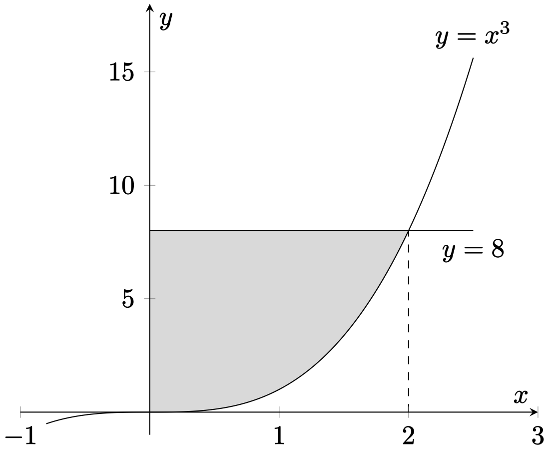
Solution. For $0 \leqslant x \leqslant 1$, the volume of a typical disc is:

$$\pi(\sqrt{x})^2 \cdot \delta x = \pi x \cdot \delta x.$$

Thus the volume of the solid is:

$$\int_0^1 \pi x \, \mathrm{d}x = \pi \left[\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

Example 9.2.2. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



Solution. For each y where $0 \leqslant y \leqslant 8$, we have the volume of a typical disc is:

$$\pi(\sqrt[3]{y})^2 \cdot \delta y.$$

Therefore, the volume of the solid is:

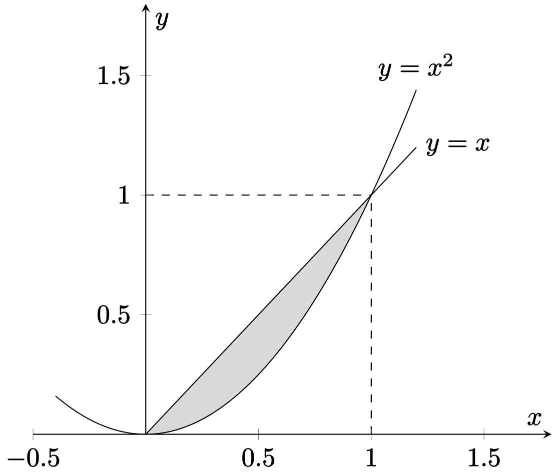
$$\int_0^8 \pi(\sqrt[3]{y})^2 \, \mathrm{d}y = \pi \left[\frac{\sqrt[3]{y^5}}{5/3} \right]_0^8 = \frac{96\pi}{5}.$$

9.2.2 The Cylindrical Shell Method

The volume of a cylindrical with radius r, height h, and thickness t is approximated by $2\pi rht$ as $t \to 0$. So, to find the volume of a solid enclosed within the x-axis rotated around the y-axis, we substitute h to be the length of a typical horizontal strip, r to be the distance from the y-axis, x, and t becomes our infinitesimal:

$$V = 2\pi \int_a^b x f(x) \, \mathrm{d}x.$$

Example 9.2.3. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

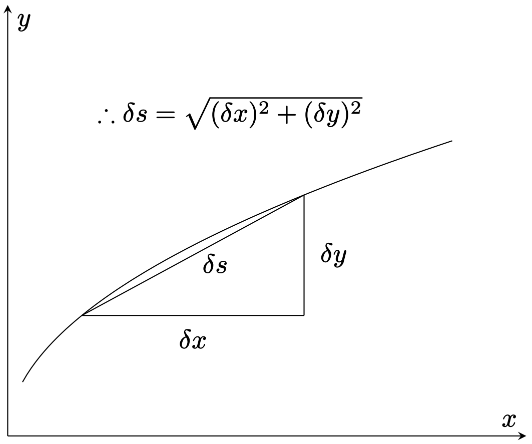


Solution. Using the cylindrical shell method, the volume of the solid is:

$$\begin{aligned}\int_0^1 2\pi x(x - x^2)\,\mathrm{d}x &= 2\pi \int_0^1 x^2 - x^3\,\mathrm{d}x \\ &= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= \frac{\pi}{6}.\end{aligned}$$

9.3 The Length of an Arc of a Curve

The length of an arc of a curve can be approximated using the Pythagoras Theorem, where the length of the curve is δs:



Therefore, the length of the arc can be approximated to be the summation of δs. This approximation gets more and more accurate as δx and δy get smaller and smaller, so passing these to the limit we have:

$$
\begin{aligned}
L &= \int_a^b \mathrm{d}s \\
&= \int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}
\end{aligned}
$$

Factoring out $\mathrm{d}x$ we have:

$$
\begin{aligned}
&= \int_a^b \sqrt{\frac{(\mathrm{d}x)^2}{(\mathrm{d}x)^2} + \frac{(\mathrm{d}y)^2}{(\mathrm{d}x)^2}}\,\mathrm{d}x \\
&= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.
\end{aligned}
$$

Likewise, for the curve defined by $x = g(y)$, where $y \in [c, d]$, the length of the arc of the curve is:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y.$$

Finally, for a parametric curve defined by:

$$y = y(t), \quad x = x(t), \quad a \leqslant t \leqslant b,$$

the length of the arc of the curve is:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

Example 9.3.1. Find the length of the arc of the curve $y = 1 + 6x^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 1$.
Solution. The length of the arc of the curve is:

$$
\begin{aligned}
\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_0^1 \sqrt{1 + \left(9x^{\frac{1}{2}}\right)^2}\,\mathrm{d}x \\
&= \int_0^1 \sqrt{1 + 81x}\,\mathrm{d}x \\
&= \left[\frac{(1+81x)^{\frac{3}{2}}}{81 \cdot \frac{3}{2}}\right]_0^1 \\
&= \frac{2}{243}\left(82^{\frac{3}{2}} - 1\right).
\end{aligned}
$$

9.4 Area of Surface of Revolution

To calculate the surface area of revolution, we must calculate the surface area of all the thin slices and add them up. The surface area of a thin slice is a frustum of a cone. This can be calculated by using our approximation of the length of an arc:

$$\delta A \approx (2\pi y)\delta s = 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\cdot\delta x.$$

Summing up all these small areas, and making δx smaller and smaller, we arrive at the following definite integral:

$$A = \int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Example 9.4.1. Find the surface area generated by revolving the curve $y = \sqrt{x+1}$ on $1 \leqslant x \leqslant 5$, about the x-axis.
Solution. The surface area is the value of the definite integral:

$$\begin{aligned}
A &= \int_1^5 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\left(\frac{1}{2\sqrt{x+1}}\right)^2}\,\mathrm{d}x \\
&= 2\pi\int_1^5 \sqrt{(x+1)\left(1+\frac{1}{4\,(x+1)}\right)}\,\mathrm{d}x \\
&= \pi\int_1^5 \sqrt{4x+5}\,\mathrm{d}x \\
&= \left[\frac{2\pi}{3}(4x+5)^{\frac{3}{2}}\right]_1^5 \\
&= \frac{2\pi}{3}(125-27) \\
&= \frac{196\pi}{3}.
\end{aligned}$$

Lecture No. 10

Improper Integrals

10.1 Unbounded Interval

Definition 10. Improper integral over an unbounded interval. If the integral:

$$\int_a^t f(x)\,\mathrm{d}x$$

exists for every number $t \geqslant a$, then we define the improper integral over an unbounded interval as follows:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.$$

It is convergent if the corresponding limit exists, and divergent if the limit does not exist.

Example 10.1.1. Determine whether $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ is convergent of divergent.

Solution. Let $t \geqslant 1$. Then:

$$\int_1^t \frac{1}{x}\,\mathrm{d}x = [\ln|x|]_1^t = \ln t,$$

which exists, and thus we have:

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty}\ln t = \infty.$$

Thus the improper limit is divergent.

Example 10.1.2. Is the improper integral $\int_0^\infty xe^{-x}\,\mathrm{d}x$ convergent? If it is, what is its value?

Solution.

$$\begin{aligned}
\int_0^\infty xe^{-x}\,\mathrm{d}x &= \lim_{t\to\infty}\int_0^t xe^{-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\left([xe^{-x}]_0^t - \int_0^t -e^{-x}\,\mathrm{d}x\right) \\
&= \lim_{t\to\infty}\left(\frac{t}{e^t}\right) + \lim_{t\to\infty}\int_0^t e^{-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\left(\frac{t}{e^t}\right) + \lim_{t\to\infty}[-e^{-x}]_0^t \\
&= \lim_{t\to\infty}\left(\frac{1}{e^t}\right) - \lim_{t\to\infty}(e^{-t}-1) \\
&= 0-(0-1) \\
&= 1.
\end{aligned}$$

$\therefore$ the series is convergent.

Theorem 22. Convergence and divergence of the integral of a function over all of $\mathbb{R}$. If both of the following integrals:

$$\int_a^\infty f(x)\,\mathrm{d}x,\ \int_{-\infty}^a f(x)\,\mathrm{d}x,$$

are convergent for some $a \in \mathbb{R}$, then we define:

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x.$$

The improper integral over $\mathbb{R}$ is said to be convergent. If either of the integrals is divergent, then the improper integral over $\mathbb{R}$ is divergent.

10.2 Infinite Discontinuity

Definition 11. Infinite discontinuity at left end-point. If f is continuous on $(a,b]$ and is discontinuous at a, then:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x,$$

if the limit exists. The improper integral is said to be convergent if the limit exists and divergent if the limit does not exist.

Example 10.2.1. Determine the value of the integral $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x$.

Solution. The function:

$$f(x) = \frac{1}{\sqrt{x-2}},$$

has an infinite discontinuity at the left endpoint of $(2,5]$. Therefore, the integral is improper, thus we evaluate it as follows:

$$\begin{aligned}
\int_2^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x &= \lim_{x\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x \\
&= \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5 \\
&= \lim_{t\to 2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right) \\
&= 2\sqrt{3}.
\end{aligned}$$

Definition 12. Infinite discontinuity at right end-point. If f is continuous on $[a, b)$ and is discontinuous at b, then:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x,$$

if the limit exists. The improper integral is said to be convergent if the limit exists and divergent if the limit does not exist.

Example 10.2.2. Evaluate $\displaystyle\int_{-1}^0 \frac{1}{x^2}\,\mathrm{d}x$.

Solution. Since $\displaystyle\lim_{x\to 0^-}\frac{1}{x^2} = \infty$, we replace the upper limit 1 by a variable $t < 0$:

$$\int_{-1}^t \frac{1}{x^2}\,\mathrm{d}x.$$

Then the improper integral is evaluated as follows:

$$\begin{aligned}
\int_{-1}^0 \frac{1}{x^2}\,\mathrm{d}x &= \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}\,\mathrm{d}x \\
&= \lim_{t\to 0^-}\left[-\frac{1}{x}\right]_{-1}^t \\
&= \lim_{t\to 0^-}\left(-\frac{1}{t}-1\right) \\
&= \infty.
\end{aligned}$$

Thus, the improper integral diverges to infinity.

Theorem 23. If f has a discontinuity at c, where $a < c < b$, and both of the following integrals are convergent:

$$\int_a^c f(x)\,\mathrm{d}x, \quad \int_c^b f(x)\,\mathrm{d}x,$$

then we define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x,$$

if either of the summands are divergent, we say that their summation is divergent.

Example 10.2.3. Evaluate $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x$.

Solution. The integrand is continuous on $(0,2)$. However, note that

$$\lim_{x\to 0^+} \frac{1}{\sqrt{2x-x^2}} = \infty,$$

and

$$\lim_{x\to 2^-} \frac{1}{\sqrt{2x-x^2}} = \infty.$$

Thus, we evaluate the given improper integral as follows:

$$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x + \int_1^2 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x.$$

We then proceed to evaluate each summand:

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x &= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x \\
&= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x \\
&= \lim_{t\to 0^+} \left[-\sin^{-1}(x-1)\right]_t^1 \\
&= \lim_{t\to 0^+} \left(-\sin^{-1}(t-1)\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

And for the second summand:

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x &= \lim_{t\to 2^-} \int_1^t \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x \\
&= \lim_{t\to 2^-} \int_1^t \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x \\
&= \lim_{t\to 2^-} \left[\sin^{-1}(x-1)\right]_1^t \\
&= \lim_{t\to 2^-} \left(\sin^{-1}(t-1)\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Thus we have:

$$\int_0^2 \frac{1}{\sqrt{2x - x^2}}\,\mathrm{d}x = \frac{\pi}{2} + \frac{\pi}{2} + \pi.$$

And we conclude that the improper integral converges to π.

Lecture No. 11

Midpoint Rule & Trapezoidal Rule

Sometimes, we do not know how to find the exact value of a definite integral. In this case, we have to settle for less by using an approximation. The approximations will come from the **Riemann sums** seen in section 4.4:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where x_i^* is any sample point in the i-th subinterval $[x_{i-1}, x_i]$, and $x_i = a + i\Delta x$, and lastly, $\Delta x = \frac{b-a}{n}$. The central idea is that different choices of x_i^* give different approximations to the definite integral.

11.1 Midpoint Rule

In the case of the midpoint rule, we choose:

$$x_i^* = \frac{1}{2}(x_i + x_{i-1}).$$

to be the midpoint of every subinterval. Thus we have:

$$\int_a^b f(x)\,\mathrm{d}x \approx M_n = \Delta x(f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})),$$

where $\overline{x_i} = \frac{1}{2}(x_i + x_{i-1})$.

11.2 Trapezoidal Rule

The intuition behind the trapezoidal rule is to add up the areas of the trapezoids created by drawing a straight line from the left and right endpoint of each segment. This is akin to averaging the left and right endpoint approximations:

$$T_n = \frac{L_n + R_n}{2}.$$

Instead of manually computing both right and left endpoint approximations, we can instead use the following expansion:

$$\begin{aligned}
T_n &= \frac{1}{2}(L_n + R_n) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right) \\
&= \frac{\Delta x}{2}\left(\sum_{i=1}^{n} (f(x_{i-1}) + f(x_i))\right) \\
&= \frac{\Delta x}{2}(f(x_0) + f(x_1) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))) \\
&= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))
\end{aligned}$$

Example 11.2.1. Use the trapezoidal rule and the midpoint rule with $n = 5$ to approximate the integral:

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x.$$

Solution. With $n = 5$, $a = 1$, $b = 2$, we have:

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 0.2.$$

The five subintervals are therefore given by the points:

$$x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8,\ x_5 = 2.$$

The trapezoidal rule gives the following:

$$\begin{aligned}
\int_1^2 \frac{1}{x}\,\mathrm{d}x &\approx \frac{0.2}{2}(f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)) \\
&\approx 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.8} + \frac{1}{2}\right) \\
&\approx 0.695635
\end{aligned}$$

The midpoints of the 5 subintervals are:

$$1.1,\ 1.3,\ 1.5,\ 1.7,\ 1.9.$$

The midpoint rule gives the following:

$$\begin{aligned}
\int_1^2 \frac{1}{x}\,\mathrm{d}x &\approx 0.2(f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)) \\
&\approx 0.2\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \\
&\approx 0.691808
\end{aligned}$$

Lecture No. 12

Error Bounds

Approximations such as the midpoint and trapezoidal rule may overestimate or underestimate the true values of definite integrals. Error bounds give an upper bound on the absolute error.

Definition 13. Error. The **error** in using an approximation is the amount that needs to be added to the approximation to make it exact:

$$\text{Error} = \text{True value} - \text{Approximation}.$$

We denote the error of an approximation as E_k where k is the name of the approximation.

Example 12.0.1. The error of the midpoint approximation is denoted as such:

$$E_M = \int_a^b f(x)\,\mathrm{d}x - M_n.$$

The following theorem gives an upper bound on the error in both the trapezoidal and midpoint rules as discussed in lecture 11.

Theorem 24. Error bounds for Trapezoidal and Midpoint Rules. Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. Then:

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}.$$

Example 12.0.2. Find n such that the midpoint rule approximation M_n approximates the following integral with an absolute error of at most 0.0001:

$$\int_1^4 \sqrt{x}\,\mathrm{d}x.$$

Solution. Let $f(x) = \sqrt{x}$. First, we must find a number K such that $|f''(x)| \leqslant K\ \forall x \in [1,4]$.

$$f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \implies |f''(x)| = \frac{1}{4}x^{-\frac{3}{2}},$$

and thus, $|f''(x)|$ is decreasing on $[1,4]$ which implies that its maximum occurs at the left endpoint $x=1$. Therefore, we choose the following:

$$K = |f''(x)| = \frac{1}{4}.$$

By the error bound theorem we have:

$$|E_M| \leqslant \frac{K(b-a)^3}{24n^2} = \frac{27}{96n^2} \leqslant 0.0001.$$

Thus the absolute error is at most 0.0001 if

$$\begin{aligned} n^2 &\geqslant \frac{27 \times 10^4}{96} \\ &= 2812.5 \\ \therefore n &\geqslant \sqrt{2812.5} \\ &= 53.033 \end{aligned}$$

So we choose $n = 54$.

Lecture No. 13

Simpson's Rule

13.1 General Formula

Simpson's Rule is a weighted average of the Trapezoidal and Midpoint Rules. It assumes that n is even and that:

$$S_n = \frac{1}{3}T_{n/2} + \frac{2}{3}M_{n/2}.$$

To define both $T_{n/2}$ and $M_{n/2}$ we consider again dividing the interval $[a, b]$ into n subintervals:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

Then, we consider the following subintervals:

$$[x_0, x_2], [x_2, x_4], \ldots, [x_{n-2}, x_n].$$

The endpoints of these subintervals $x_0, x_2, \ldots, x_{n-2}, x_n$ are used to compute $T_{n/2}$:

$$\begin{aligned} T_{n/2} &= \frac{1}{2}\frac{b-a}{n/2}(f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)) \\ &= \Delta x(f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)) \end{aligned}$$

The midpoints of the subintervals $x_1, x_3, \ldots, x_{n-3}, x_{n-1}$ are used to compute $M_{n/2}$:

$$\begin{aligned} M_{n/2} &= \frac{b-a}{n/2}(f(x_1) + f(x_3) + \cdots + f(x_{n-3}) + f(x_{n-1})) \\ &= \Delta x(2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-3}) + 2f(x_{n-1})) \end{aligned}$$

Combining the two we get:

$$S_n = \frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

Example 13.1.1. Use Simpson's Rule with $n = 8$ to find $\displaystyle\int_2^4 \sqrt{1 + x^3}\,\mathrm{d}x$.

Solution. We have:

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{8} = 0.25,$$

and therefore we have:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
2	2.25	2.5	2.75	3	3.25	3.5	3.75	4

Now, let:

$$f(x) = \sqrt{1+x^3}.$$

Thus:

$$\begin{aligned}
S_8 &= \frac{1}{3}(0.25)(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) \\
&\quad + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)) \\
&= \frac{1}{12}\Big(\sqrt{1+2^3} + 4\sqrt{1+2.25^3} + 2\sqrt{1+2.5^3} + 4\sqrt{1+2.75^3} \\
&\quad + 2\sqrt{1+3^3} + 4\sqrt{1+3.25^3} + 2\sqrt{1+3.5^3} \\
&\quad + 4\sqrt{1+3.75^3} + \sqrt{1+4^3}\Big) \\
&\approx \frac{1}{12}(3 + 4(3.52003) + 2(4.07738) + 4(4.66871) + 2(5.2915) \\
&\quad + 4(5.94375) + 2(6.62382) + 4(7.33037) + 8.06226) \\
&\approx 10.74159.
\end{aligned}$$

As required.

13.2 Error Bounds

Error bounds for the trapezoidal and midpoint rule rely on finding the second derivative of the integrand, however, error bounds for Simpson's rule depends on the fourth derivative of the function.

Theorem 25. Error Bound for Simpson's Rule. Suppose that $|f^{(4)}(x)| \leqslant K$ for all $a \leqslant x \leqslant b$. If E_S is the error involved in using the Simpson's Rule, then:

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}.$$

Example 13.2.1. Find n such that S_n has an absolute error of at most 10^{-6} for the following:

$$S_n \approx \int_1^3 \frac{1}{x}\,\mathrm{d}x.$$

Solution. Let $f(x) = \dfrac{1}{x}$, then:

$$f^{(4)}(x) = 24x^{-5},$$

which is decreasing, so its maximum on $[1,3]$ is $f^{(4)}(1) = 24$. Thus we choose $K = 24$, and by using the error bound theorem we see that:

$$|E_S| \leqslant \frac{24(3-1)^5}{180n^4} = \frac{64}{15n^4}.$$

To ensure that the error is at most 10^{-6}, we choose n so that:

$$|E_S| \leqslant \frac{64}{15n^4} \leqslant 10^{-6}.$$

Which is true iff:

$$\begin{aligned} n^4 &\geqslant 10^6 \left(\frac{64}{15}\right) \\ \therefore n &\geqslant \left(10^6 \left(\frac{64}{15}\right)\right)^{\frac{1}{4}} \\ &\approx 45.45. \end{aligned}$$

Thus, we should take $n = 46$ which works out perfectly since our choice for n has to be even for Simpson's Rule.

Lecture No. 14

The Integral Test

14.1 Intuition Behind Integral Test

The **integral test** is a type of convergence test. A convergence test is a test we can attempt to apply on a series to better determine its convergence. We see how the test works using an example.

Example 14.1.1. Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

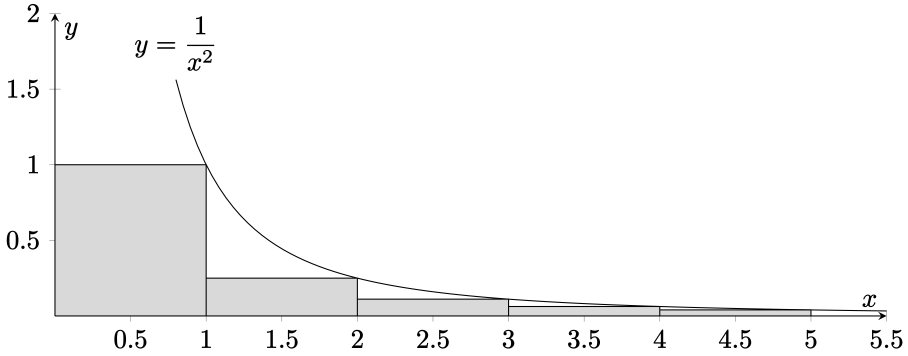
It is known that the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ is convergent since:

$$\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^t = 1.$$

Yet, we note that:

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leqslant \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = 1.$$

As seen in the following diagram:



Therefore, the k-th partial sums of the series are bounded:

$$s_k = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leqslant 1 + \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1 + 1 = 2.$$

On top of that, the sequence of partial sums $\{s_k\}$ is increasing, since:

$$s_{k+1} = s_k + \frac{1}{(k+1)^2} > s_k.$$

Hence, by the **monotonic convergence theorem**, the sequence $\{s_k\}$ is convergent, and thus the series is convergent.

Theorem 26. The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1,\infty)$, and let $a_n = f(n)$. The following implications hold true:

$$\int_1^\infty f(x)\,\mathrm{d}x \text{ is convergent } \implies \sum_{n=1}^\infty a_n \text{ is convergent.}$$

$$\int_1^\infty f(x)\,\mathrm{d}x \text{ is divergent } \implies \sum_{n=1}^\infty a_n \text{ is divergent.}$$

Note. When using the integral test, it is not necessary that the series or the integral starts at $n = 1$. It is also not necessary that f is always decreasing. It is necessary for f to be *ultimately* decreasing, that is, f is decreasing $\forall x \geqslant N$, where $N \in \mathbb{Z}$.

Example 14.1.2. Test the series $\displaystyle\sum_{n=1}^\infty \frac{1}{1+n^2}$ for convergence.

Solution. The function $f(x) = \dfrac{1}{1+x^2}$ is continuous, positive, and decreasing on $[1,\infty)$. $\therefore$ we can use the integral test:

$$\begin{aligned}
\int_1^\infty \frac{1}{1+x^2}\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t \\
&= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\end{aligned}$$

Thus, by the integral test, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{1+n^2}$ is convergent.

14.2 p-series

An important class of series consists of p-series. The convergence of a p-series can be determined using the integral test.

Theorem 27. p-series. A p-series is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is some fixed real number. It is convergent when $p > 1$, and divergent when $p \leqslant 1$.

Proof. We consider different cases depending on p. Say:

$$p < 0.$$

Then the term $\frac{1}{n^p}$ can be arbitrarily large for sufficiently large n, that is:

$$\lim_{n\to\infty} \frac{1}{n^p} = \infty,$$

so, by the n-th term divergence test, the series is divergent. Now, consider the case where:

$$p = 0.$$

In this case, the term $\frac{1}{n^p} = 1$ for all values of n, and so by the n-th term divergence test, the series is divergent. For the case where:

$$p = 1,$$

we have $\frac{1}{n^p} = \frac{1}{n}$. Therefore, the p-series becomes the harmonic series which is known to be divergent. Finally, for the case where:

$$p \neq 1 \wedge p > 0,$$

we have:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^t x^{-p}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t \\
&= \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right)
\end{aligned}$$

Now, consider the case where $p > 1$, then $p - 1 > 0$, and so $t^{1-p} = \frac{1}{t^{p-1}} \to 0$ as $t \to \infty$. Therefore:

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right) = \frac{1}{1-p}(0-1) = \frac{1}{p-1}.$$

By the integral test, the corresponding p-series is convergent. Now, consider the case where $0 < p < 1$, then $1 - p > 0$, and so $t^{1-p} \to \infty$ as $t \to \infty$. Therefore,

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right) = \infty.$$

By the integral test, the corresponding p-series is divergent. □

Example 14.2.1. Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is convergent.

Solution. Consider the function:

$$f(x) = \frac{\ln x}{x}.$$

Clearly, $f(x)$ is positive and continuous for all $x > 1$. To show that f is decreasing, we compute the derivative:

$$f'(x) = \frac{\left(\frac{1}{x}\right) x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

Thus, $f'(x) < 0$ for all $\ln x > 1$, therefore f is decreasing for all $x > e$ and $x \geqslant 3$. In view of the integral test, we check the improper integral:

$$\begin{aligned}
\int_3^{\infty} \frac{\ln x}{x} \, \mathrm{d}x &= \lim_{t \to \infty} \int_3^t \frac{\ln x}{x} \, \mathrm{d}x \\
&= \lim_{t \to \infty} \left[\frac{1}{2} (\ln x)^2 \right]_3^t \\
&= \lim_{t \to \infty} \left(\frac{(\ln t)^2}{2} - \frac{1}{2} (\ln 3)^2 \right) = \infty.
\end{aligned}$$

By the integral test, the series is divergent.

Lecture No. 15

The Comparison Tests

15.1 The Series Comparison Test

If we have a series whose terms are smaller than those of a known convergent series, then our series is also convergent. If we have a series whose terms are bigger than those of a known divergent series, then our series is also divergent.

Theorem 28. The Series Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Then the following implications hold true:

$$\sum b_n \text{ is convergent } \wedge \ a_n \leqslant b_n \ \forall n \implies \sum a_n \text{ is convergent}$$
$$\sum b_n \text{ diverges to } \infty \ \wedge \ a_n \geqslant b_n \ \forall n \implies \sum a_n \text{ diverges to } \infty$$

Proof. Let:

$$s_k = \sum_{n=1}^{k} a_n, \ t_k = \sum_{n=1}^{k} b_n,$$

denote the k-th partial sums of the series. Now, suppose $\sum b_n$ is convergent, and $a_n \leqslant b_n \ \forall n$. The sequence $\{s_k\}$ and $\{t_k\}$ are increasing, since each term in a_n and b_n are positive. Since $a_n \leqslant b_n \ \forall n$, we have:

$$s_k = \sum_{n=1}^{k} a_n \leqslant \sum_{n=1}^{k} b_n = t_k.$$

But $\sum b_n$ is convergent, that is, $\lim_{k\to\infty} t_k = t$ for some $t \in \mathbb{R}$. This implies that $t_k \leqslant t$ for all k since $\{t_k\}$ is increasing. So:

$$s_k \leqslant t_k \leqslant t.$$

We have shown that $\{s_k\}$ is increasing and bounded. By the monotonic convergence theorem, $\lim_{k\to\infty} s_k$ exists, that is:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = s,$$

for some $s \in \mathbb{R}$. Therefore the series is convergent. Now, suppose $\sum b_n$ diverges to ∞, and $a_n \geqslant b_n$ $\forall n$. Since $a_n \geqslant b_n$ $\forall n$, we have:

$$s_k \geqslant t_k.$$

Since $\lim_{k\to\infty} t_k = \infty$, we deduce that:

$$\lim_{k\to\infty} s_k = \infty.$$

Therefore, the series $\sum a_n$ also diverges to ∞. □

Example 15.1.1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{100}{2n^2+5n+4}$ is convergent.

Solution. Set $a_n = \dfrac{100}{2n^2+5n+4}$. Note that:

$$a_n = \frac{100}{2n^2+5n+4} < \frac{100}{2n^2} = \frac{50}{n^2}.$$

Set $b_n = \dfrac{50}{n^2}$. We know that:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{50}{n^2} = 50 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is convergent since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p > 1$. Therefore, by the series comparison test, the series is convergent.

15.2 The Limit Comparison Test

Sometimes, the following limit version of the series comparison test is easier to apply.

Theorem 29. The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Then the following implications hold true:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \implies \begin{pmatrix}\text{both series}\\ \text{converge}\end{pmatrix} \veebar \begin{pmatrix}\text{both series}\\ \text{diverge to } \infty\end{pmatrix}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \wedge \sum b_n \text{ converges} \implies \sum a_n \text{ converges}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \wedge \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}$$

Proof. We first prove the first implication. Let m and M be positive numbers such that $m < c < M$. Since $\dfrac{a_n}{b_n}$ converges to c, for a large enough n, we would have:

$$m < \frac{a_n}{b_n} < M, \ \forall n > N.$$

Thus,

$$mb_n < a_n < Mb_n, \ \forall n > N.$$

Now, if $\sum b_n$ converges, then so does $\sum Mb_n$. Therefore, by the series comparison test, $\sum a_n$ is convergent since $a_n < Mb_n$.

Then, if $\sum a_n$ converges, then by the series comparison test, $\sum mb_n$ converges since $mb_n < a_n$. This implies that $\sum b_n$ also converges. Thus we have proved the following:

$$\sum a_n \text{ converges} \iff \sum b_n \text{ converges.}$$

Now, if $\sum b_n$ diverges to ∞, then so does $\sum mb_n$. Then, by the series comparison test, $\sum a_n$ diverges to ∞ since $a_n > mb_n$.

Then, if $\sum a_n$ diverges to ∞, then by the series comparison test, $\sum Mb_n$ diverges to ∞ since $Mb_n > a_n$. This implies that $\sum b_n$ also diverges to ∞. Thus we have also proved the following:

$$\sum a_n \text{ diverges to } \infty \iff \sum b_n \text{ diverges to } \infty.$$

For the second implication in the theorem, we have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0,$$

so, we let M be a positive number. Thus, there exists a number N such that:

$$0 < \frac{a_n}{b_n} < M, \ \forall n > N.$$

Therefore, we have:

$$0 < a_n < Mb_n, \ \forall n > N.$$

Since $\sum b_n$ converges, then so does $\sum Mb_n$. So, by the series comparison test, $\sum a_n$ is convergent.

For the last implication, we have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty,$$

so, we let M be a positive number. Thus there exists a number N such that:

$$M < \frac{a_n}{b_n} < \infty, \ \forall n > N.$$

Therefore, we have:

$$Mb_n < a_n < \infty, \ \forall n > N.$$

Since $\sum b_n$ diverges to ∞, then so does $\sum Mb_n$. So, by the series comparison test, $\sum a_n$ diverges to ∞. □

Example 15.2.1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence.

Solution. We use the limit comparison test with:

$$a_n = \frac{1}{2^n - 1}, \quad b_n = \frac{1}{2^n}.$$

Thus we have:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1/(2^n-1)}{1/2^n}\\ &= \lim_{n\to\infty}\frac{2^n}{2^n-1}\\ &= \lim_{n\to\infty}\frac{1}{1-1/2^n}=1>0.\end{aligned}$$

Since b_n is convergent, the series a_n is also convergent by the limit comparison test.

Example 15.2.2. Determine if the series $\sum_{n=1}^{\infty}\frac{2n^2+3n}{\sqrt{5+n^5}}$ is convergent.

Solution. The dominant part of the numerator is $2n^2$, and the dominant part of the denominator is $\sqrt{n^5}=n^{\frac{5}{2}}$. This suggests taking:

$$a_n=\frac{2n^2+3n}{\sqrt{5+n^5}},\quad b_n=\frac{2n^2}{n^{\frac{5}{2}}}=\frac{2}{\sqrt{n}}.$$

Thus we have:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{2n^2+3n}{\sqrt{5+n^5}}\cdot\frac{n^{\frac{1}{2}}}{2}\\ &= \lim_{n\to\infty}\frac{2n^{\frac{5}{2}}+3n^{\frac{3}{2}}}{2\sqrt{5+n^5}}\\ &= \lim_{n\to\infty}\frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}}\\ &= \frac{2+0}{2\sqrt{0+1}}=1.\end{aligned}$$

Since the sum of b_n is divergent(p-series with $p=\frac{1}{2}$), the limit comparison test tells us that a_n is also divergent.

Lecture No. 16

Absolute & Conditional Convergence

16.1 Absolute Convergence

Definition 14. Absolute Convergence. A series $\sum a_n$ is **absolutely convergent** if the series:

$$\sum |a_n|$$

converges.

Example 16.1.1. The series

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because the corresponding series with absolute values is the convergent p-series:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

The following result states that if the series of absolute values converges, then the original series converges.

Theorem 30. Absolute Convergence implies Convergence. If $\sum a_n$ is absolutely convergent, then $\sum a_n$ converges.

Proof. We have:

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|.$$

Thus, $\sum(a_n + |a_n|)$ is a series with positive terms (dropping all 0's).

Now, suppose that $\sum a_n$ is absolutely convergent, that is, $\sum |a_n|$ converges. Then, $\sum 2|a_n|$ also converges. Then, by the series comparison test, we can say that $\sum(a_n + |a_n|)$ is convergent.

Our original series is the difference of two convergent series and hence it converges:

$$\sum a_n = \sum((a_n + |a_n|) - |a_n|) = \underbrace{\sum(a_n + |a_n|)}_{\text{convergent}} - \underbrace{\sum |a_n|}_{\text{convergent}} .$$

□

Example 16.1.2. Determine whether the series:

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.
Solution. The series has both positive and negative terms, but it is not alternating: The first term is positive, yet the next three are negative. The series of absolute values is:

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}.$$

Since $|\cos n| \leqslant 1$ for all n, we have:

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Therefore, by the series comparison test:

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

is also convergent. This implies that the series in question is absolutely convergent, and therefore is convergent itself.

Example 16.1.3. Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converge absolutely?
Solution. We apply the integral test to the series of absolute values:

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Using the substitution $u = \ln x$, $\mathrm{d}u = x^{-1}\,\mathrm{d}x$, we have:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x}\,\mathrm{d}x &= \int_{\ln 2}^{\infty} \frac{1}{u}\,\mathrm{d}u \\ &= \lim_{t \to \infty} \int_{\ln 2}^{t} \frac{1}{u}\,\mathrm{d}u \\ &= \lim_{t \to \infty} (\ln t - \ln(\ln 2)) \\ &= \infty. \end{aligned}$$

Therefore, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, so $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is not absolutely convergent.

16.2 Conditional Convergence

Definition 15. Conditional Convergence. A series $\sum a_n$ is called **conditionally convergent** if it converges but $\sum |a_n|$ diverges.

The following theorem provides a test for conditional convergence if a series is not absolutely convergent.

Theorem 31. Alternating Series Test. Let $\{a_n\}$ be a decreasing positive sequence that converges to 0:

$$a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant \cdots \geqslant 0, \quad \lim_{n\to\infty} a_n = 0.$$

Then the following alternating series converges:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots .$$

Note. This alternating series starts with a positive term and then alternates. This result also applies if it starts with a negative term, which then alternates.

Note. The proof of this theorem is beyond the scope for this course.

Example 16.2.1. Show that $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent.

Solution. The terms:

$$a_n = \frac{1}{\sqrt{n}},$$

form a decreasing sequence that converges to 0. The alternating series test implies that S is convergent. However, S is only conditionally convergent because the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p-series.

Example 16.2.2. Is the series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ convergent?

Solution. Let:

$$a_n = \frac{n^2}{n^3+1}.$$

In view of the alternating series test, we want to show that $\{a_n\}_{n=1}^{\infty}$ is decreasing and $a_n \to 0$. Let's consider the related function $f(x)$, where $f(n) = a_n$. Its derivative is:

$$f(x) = \frac{x^2}{x^3+1} \xrightarrow{\text{differentiation}} f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}.$$

Since we are considering only positive x, we see that:

$$f'(x) < 0 \iff 2 - x^3 < 0 \iff x > 2^{\frac{1}{3}}.$$

Thus, f is strictly decreasing on the interval $(2^{\frac{1}{3}}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $a_{n+1} < a_n$ for all $n \geqslant 2$. On the other hand, it is clear that:

$$a_2 = \frac{2^2}{2^3+1} = \frac{4}{9} < \frac{1}{2} = a_1.$$

So the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing. The limit of $\{a_n\}$ is readily verified:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$$

Hence, the given series is convergent by the alternating series test.

Lecture No. 17

The Ratio & Root Tests

The convergence tests outlined so far cannot be easily applied to series containing factorial terms or n-th powers. For this, we will need the following tests.

17.1 The Ratio Test

Theorem 32. Ratio Test. Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then the following implications hold true:

$$\begin{aligned} \rho < 1 &\implies \sum a_n \text{ converges absolutely} \\ \rho > 1 \vee \rho = \infty &\implies \sum a_n \text{ diverges} \\ \rho = 1 &\implies \text{Ratio test is inconclusive} \end{aligned}$$

Proof. First, we consider the case where $\rho < 1$. We choose a number r such that:

$$\rho < r < 1.$$

Since $\left|\frac{a_{n+1}}{a_n}\right|$ converges to ρ, there exists a number N such that:

$$\left|\frac{a_{n+1}}{a_n}\right| < r, \ \forall n > N.$$

That is:

$$|a_{n+1}| < r|a_n|, \ \forall n > N.$$

In particular, we have:

$$\begin{aligned} |a_{N+1}| &< r|a_N| \\ |a_{N+2}| &< r|a_{N+1}| < r^2|a_N| \\ &\vdots \\ |a_{N+k}| &< r^k|a_N|. \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=N}^{\infty}|a_n| &= \sum_{k=0}^{\infty}|a_{N+k}| \\ &\leqslant \sum_{k=0}^{\infty} r^k|a_N| \\ &= |a_N|\sum_{k=0}^{\infty} r^k.\end{aligned}$$

The geometric series on the right hand side converges because $0 < r < 1$, and so the left hand side converges by the comparison test. Thus the series $\sum a_n$ converges absolutely.

Secondly, we consider the case where $\rho > 1 \vee \rho = \infty$. We choose a number r such that $1 < r < \rho$. Arguing as before with inequalities reversed, we conclude that there exists an integer N such that:

$$|a_{N+k}| \geqslant r^k|a_N|,\ \forall k \geqslant 0.$$

Since r^k tends to ∞, we see that the term a_{N+k} does not tend to 0. Thus, by the n-th term test for divergence, we can conclude that the series $\sum a_n$ is divergent. ☐

Example 17.1.1. Consider the series $\sum_{n=1}^{\infty} n^2$ and $\sum_{n=1}^{\infty} n^{-2}$. Find ρ for each.

Solution. Let $a_n = n^2$. Then:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{n^2} \\ &= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \\ &= \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) \\ &= 1.\end{aligned}$$

On the other hand, let $b_n = n^{-2}$. Then:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| \\ &= \lim_{n\to\infty}\frac{n^2}{(n+1)^2} \\ &= \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} \\ &= \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} \\ &= 1.\end{aligned}$$

Thus, $\rho = 1$ in both cases. However, one series converges and one diverges, therefore, the ratio test is inconclusive when $\rho = 1$.

Example 17.1.2. Using the ratio test, show that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Solution. We compute the limit ρ. Let $a_n = \frac{1}{n!}$. Then:

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1}.$$

Thus we have:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

Example 17.1.3. Apply the ratio test to determine if $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

Solution. Let $a_n = \frac{n^2}{2^n}$. We have:

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\ &= \frac{1}{2}\left(\frac{n^2+2n+1}{n^2}\right) \\ &= \frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \\ \therefore \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{2}. \end{aligned}$$

Since $\rho < 1$, the series converges.

17.2 The Root Test

For some series, it is more convenient to use the following root test, based on the limit of the n-th roots $\sqrt[n]{a_n}$ rather than the ratio of consecutive terms.

Definition 16. Root Test. Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then the following implications hold true:

$$\begin{aligned} L < 1 &\implies \sum a_n \text{ converges absolutely} \\ L > 1 \vee L = \infty &\implies \sum a_n \text{ diverges} \\ L = 1 &\implies \text{Root test is inconclusive} \end{aligned}$$

Proof. First, we consider the case where $L < 1$. We choose a number r such that:

$$L < r < 1.$$

Now, because we have chosen r such that $L < r$, there is some N such that if $n \geqslant N$ we will have:

$$\sqrt[n]{|a_n|} < r \implies |a_n| < r^n.$$

Now, the series:

$$\sum_{n=1}^{\infty} r^n$$

is a geometric series and because $0 < r < 1$ we in fact know that it is a convergent series. Thus, by the comparison test we conclude that:

$$\sum_{n=N}^{\infty} |a_n|$$

is convergent. However, since:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|,$$

we know that the left hand side is convergent since it is the sum of two convergent sums.

Next, we assume that $L > 1$, and because of that, we know that there must be some N such that if $n \geqslant N$ we have:

$$\sqrt[n]{|a_n|} > 1 \implies |a_n| > 1^n = 1.$$

However, if $|a_n| > 1$ for all $n \geqslant N$ then we know that:

$$\lim_{n\to\infty} |a_n| \neq 0 \implies \lim_{n\to\infty} a_n \neq 0.$$

Therefore, the sum is divergent. $\square$

Example 17.2.1. Determine whether $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges.

Solution. Let $a_n = \left(\dfrac{n}{2n+3}\right)^n$. Then:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}.$$

Since $L < 1$, the series converges.

Lecture No. 18

Power Series

18.1 The Power Series

A differentiable function can be expressed as an 'infinite polynomial' called a power series.

Definition 17. Power Series. A **power series** in x is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where x is a variable, and the c_n's are constants called the **coefficients** of the series.

Note. A power series resembles a polynomial in x. The only difference is that f has infinitely many terms, while a polynomial has only finitely many terms.
Example 18.1.1. The geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots ,$$

is an example of a power series which converges when $-1 < x < 1$ and diverges when $|x| \geqslant 1$.

Definition 18. Power series at $x = a$. More generally, a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a **power series at** $x = a$.

Example 18.1.2. For what values of x is the series $\sum\limits_{n=0}^{\infty} n!x^n$ convergent?

Solution. Clearly, the series converges when $x = 0$. Suppose $x \neq 0$. We use the ratio test with $a_n = n!x^n$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty.$$

By the ratio test, the series diverges whenever $x \neq 0$. So the series converges only when $x = 0$.

18.2 Radius of Convergence

Theorem 33. Radius of Convergence. For any given power series:

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$, and diverges if $|x-a| > R$.

The number R is called the **radius of convergence** of the power series. In the first case, $R = 0$, and in the second case, $R = \infty$.

Example 18.2.1. Find the radius of convergence and the interval of convergence of the series:

$$\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Solution. Let $a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$. Then:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\ &= \lim_{n\to\infty}\left|(-3x)\sqrt{\frac{n+1}{n+2}}\right| \\ &= \lim_{n\to\infty} 3\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}|x| \\ \therefore \rho &= 3|x|.\end{aligned}$$

By the ratio test, we see that the series converges if $3|x| < 1$ and diverges if $3|x| > 1$. This means that the radius of convergence, $R = \frac{1}{3}$. However, the ratio test provides no information regarding the cases where $3|x| = 1$, that is, when $x = \pm\frac{1}{3}$. We consider these cases separately.

If $x = -\frac{1}{3}$, then the series becomes:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}},$$

which diverges because:

$$\frac{1}{\sqrt{n+1}} \geqslant \frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}},$$

and since $\dfrac{1}{\sqrt{n}}$ diverges, by the comparison test, the series also diverges.

If $x = \frac{1}{3}$, then the series becomes:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}},$$

which converges by the alternating series test. In summary, the interval of convergence of the given power series is $(-\frac{1}{3}, \frac{1}{3}]$.

Lecture No. 19

Representation of Functions as Power Series

19.1 Fitting to a Geometric Series

Consider the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

We know that this series is convergent for all $|x| < 1$. We can use this series to express a given function as a sum of a power series.
Example 19.1.1. Express the function:

$$f(x) = \frac{1}{1+x^2},$$

as the sum of a power series in x and find the interval of convergence.
Solution. Replace x by $-x^2$ in the standard geometric series to get:

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty}(-x^2)^n \\ &= 1 - x^2 + x^4 - x^6 + x^8 - \cdots\end{aligned}$$

Since this is a geometric series, it converges when:

$$|-x^2| < 1 \iff |x^2| < 1 \iff |x| < 1.$$

Therefore, the interval of convergence is $(-1, 1)$.
Example 19.1.2. Find a power series representation of:

$$f(x) = \frac{1}{3x+1},$$

at $x = 1$, and determine its interval of convergence.

Solution. Again, we use the standard geometric series:

$$\begin{aligned}\frac{1}{3x+1} &= \frac{1}{3(x-1)+4}\\ &= \frac{1}{4\left(1-\left(-\frac{3(x-1)}{4}\right)\right)}\\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{3(x-1)}{4}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{4^{n+1}}(x-1)^n.\end{aligned}$$

The series converges when:

$$\left|-\frac{3(x-1)}{4}\right| < 1 \iff |x-1| < \frac{4}{3}.$$

Thus the interval of convergence is $\left(-\frac{1}{3}, \frac{7}{3}\right)$.

Example 19.1.3. Find a power series representation of $\dfrac{x^3}{x+2}$ in x.

Solution. We first find the power series representation of $\dfrac{1}{x+2}$ using the standard geometric series:

$$\begin{aligned}\frac{1}{x+2} &= \frac{1}{2}\cdot\frac{1}{1-(-\frac{x}{2})}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n.\end{aligned}$$

Multiplying by x^3:

$$\begin{aligned}\frac{x^3}{x+2} &= x^3\cdot\frac{1}{x+2}\\ &= x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}\\ &= \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n \qquad \text{(after reindexing)}\\ &= (-4)\sum_{n=3}^{\infty}\left(-\frac{x}{2}\right)^n.\end{aligned}$$

Which converges iff:

$$\left|-\frac{x}{2}\right| < 1 \iff |x| < 2.$$

Thus the interval of convergence is $(-2, 2)$.

19.2 Integration & Differentiation of Power Series

Theorem 34. Term-by-term Differentiation & Integration of Power Series. Suppose that the power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has a radius of convergence $R > 0$. Then $f(x)$ is differentiable on the interval $(a-R, a+R)$, and its derivative and anti derivative may be computed term by term. More precisely,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Moreover, the series above both have the same radius of convergence R.

Example 19.2.1. Prove that:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

for $-1 < x < 1$.

Solution. Note that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}.$$

Thus, we obtain the result by differentiating the geometric series term by term for $|x| < 1$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + x^2 + x^3 + \cdots\right)\\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots\end{aligned}$$

The above expansion is valid for $|x| < 1$.

Example 19.2.2. Prove that for $-1 < x < 1$:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Solution. First, we substitute $-x^2$ into x in the geometric series to obtain:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots.$$

Since the geometric series has a radius of convergence $R = 1$, this expansion is valid for $\left|x^2\right| < 1$, that is $|x| < 1$. Now, recall that:

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,\mathrm{d}x.$$

Thus, we integrate this series term by term to get:

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2}\,\mathrm{d}x \\ &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right)\mathrm{d}x \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

Where C is a constant. When we set $x = 0$, we obtain $\tan^{-1} 0 = 0 = C$. This proves the result as required.

Lecture No. 20

Taylor & Maclaurin Series

The Taylor & Maclaurin Series are general methods for finding power series representations of functions.

20.1 Taylor & Maclaurin Series

Suppose that $f(x)$ has a power series expansion centred at $x = a$ that is valid for all x in an interval $(a - R, a + R)$ with $R > 0$. If such a power series expansion did exist, it would have the following form:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where $|x - a| < R$. Then, our goal is simply to determine the coefficients $c_0, c_1, c_2, \ldots$

- To find c_0: Substitute $x = a$ into the power series, getting:

  $$c_0 = f(a).$$

- To find c_1: Differentiate the power series term by term:

  $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

  Then, substitute $x = a$ into the above to give:

  $$c_1 = f'(a).$$

- To find c_2: Differentiate the power series term by term another time:

  $$f^{(2)}(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots$$

  Substitute $x = a$ into the above getting:

  $$c_2 = \frac{1}{2} f^{(2)}(a) = \frac{1}{2!} f^{(2)}(a).$$

- To find c_3: Differentiate the power series term by term another time:

  $$f^{(3)}(x) = 3 \cdot 2c_2 + 4 \cdot 3 \cdot 2c_3(x-a) + 5 \cdot 4 \cdot 3c_4(x-a)^2 + \cdots$$

  Substitute $x = a$ into the above getting:

  $$c_3 = \frac{1}{3 \cdot 2} f^{(3)}(a) = \frac{1}{3!} f^{(3)}(a).$$

Clearly, there is a general pattern. Solving the equation for the n-th coefficient, we get:

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Theorem 35. Uniqueness of the Power Series Expansion. If f has a power series expansion at $x = a$, that is, if:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ |x-a| < R, \ R > 0,$$

then its coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This power series is called the **Taylor Series of $f(x)$ centred at $x = a$.** In the special case where $a = 0$, the Taylor series is also called the **Maclaurin Series**.

Example 20.1.1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.
Solution. Note that $f^{(n)}(x) = e^x$ for all n. Therefore:

$$f^{(n)}(0) = 1, \ \forall n \geqslant 0.$$

Therefore, the Maclaurin series of f is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence, we let $a_n = \dfrac{x^n}{n!}$. Then:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

So, by the ratio test, the series converges for all x, and the radius of convergence is $R = \infty$.
Note. This example only tells us that **if** there exists a power series for e^x, then it is equal to the Maclaurin series as calculated.

20.2 Remainder & Error Bounds of Power Series

20.2.1 Taylor's Theorem & The Remainder Polynomial

To show that a function has a power series representation, we need to study the convergence of Taylor series. Let us consider the n-th degree Taylor polynomial of f at a:

$$\begin{aligned}
T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\end{aligned}$$

Note that the n-th degree Taylor polynomial is just the sum of the first $(n+1)$ terms in the corresponding Taylor series. Therefore, $f(x)$ can be represented by its Taylor series if:

$$f(x) = \lim_{n\to\infty} T_n(x).$$

We denote the n-th remainder of f at a by:

$$R_n(x) = f(x) - T_n(x).$$

It follows that:

$$f(x) = T_n(x) + R_n(x).$$

So if we can show that the remainder vanishes, that is:

$$\lim_{n\to\infty} R_n(x) = 0 \implies \lim_{n\to\infty} T_n(x) = f(x).$$

We have therefore proved the following theorem.

Theorem 36. Remainder Polynomial tends to 0. If:

$$\lim_{n\to\infty} R_n(x) = 0, \ \forall |x-a| < R,$$

then f is equal to its Taylor series on the interval $|x-a| < R$.

Next, we study $R_n(x)$, known as Taylor's Theorem.

Theorem 37. Taylor's Theorem. Assume that $f^{(n+1)}(x)$ exists and is continuous. Then:

$$R_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,\mathrm{d}u.$$

Proof. Taylor's Theorem. Set:

$$I_n = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,\mathrm{d}u.$$

For $n = 0$, we have by definition that:

$$R_0(x) = f(x) - f(a).$$

On the other hand,

$$I_0(x) = \frac{1}{0!}\int_a^x (x-u)^0 f^{(0+1)}(u)\,\mathrm{d}u = \int_a^x f'(x)(u)\,\mathrm{d}u = f(x) - f(a),$$

by the fundamental theorem of calculus. Hence, we have proved that:

$$R_0(x) = I_0(x).$$

To prove the formula for $n > 0$, we apply integration by parts to $I_n(x)$ with:

$$U = \frac{(x-u)^n}{n!}, \ V' = f^{(n+1)}(u).$$

Then:

$$U' = -\frac{(x-u)^{n-1}}{(n-1)!}, \ V = f^{(n)}(u).$$

So we have:

$$\begin{aligned}
I_n(x) &= \int_a^x UV' \, \mathrm{d}u \\
&= [UV]_a^x - \int_a^x U'V \, \mathrm{d}u \\
&= \left[\frac{1}{n!}(x-u)^n f^{(n)}(u)\right]_a^x - \int_a^x -\frac{(x-u)^{n-1}}{(n-1)!} f^{(n)}(u) \, \mathrm{d}u \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) + I_{n-1}(x).
\end{aligned}$$

The result can be rewritten as:

$$I_{n-1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + I_n(x).$$

Now, apply the recurrence relation n times:

$$\begin{aligned}
f(x) &= f(a) + I_0(x) \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + I_1(x) \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + I_2(x) \\
&\vdots \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + I_n(x) \\
&= T_n(x) + I_n(x).
\end{aligned}$$

This shows that $I_n(x) = f(x) - T_n(x)$ which, by definition must be equal to $R_n(x)$. □

20.2.2 Error Bound for Taylor Series

Theorem 38. Error Bound for Taylor Series. Let M be a number such that $|f^{(n+1)}(x)| \leqslant M$, $\forall |x-a| \leqslant d$. Then:

$$|R_n(x)| \leqslant M\frac{|x-a|^{n+1}}{(n+1)!}, \quad |x-a| \leqslant d.$$

Proof. Error Bound for Taylor Series. Assume that $a \leqslant x \leqslant a+d$. Then, since $|f^{(n+1)}(u)| \leqslant M$ for all $a \leqslant u \leqslant x$ we have:

$$\begin{aligned}|R_n(x)| &= \left|\frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,\mathrm{d}u\right| \\ &\leqslant \frac{1}{n!}\int_a^x |(x-u)^n f^{(n+1)}(u)|\,\mathrm{d}u\end{aligned}$$

And since $a \leqslant u \leqslant x$, we ignore the absolute value sign:

$$\begin{aligned}&\leqslant \frac{M}{n!}\int_a^x (x-u)^n\,\mathrm{d}u \\ &= \frac{M}{n!}\left[\frac{-(x-u)^{n+1}}{n+1}\right]_a^x \\ &= M\frac{|x-a|^{n+1}}{(n+1)!}.\end{aligned}$$

□

Example 20.2.1. Prove that $f(x) = e^x$ is equal to its Maclaurin Series.
Solution. It is enough to show that the remainder polynomial $R_n(x)$ converges to 0 for all x. First, pick an arbitrary positive number d. Note that:

$$|f^{(n+1)}(x)| = e^x \leqslant e^d, \ \forall |x| \leqslant d.$$

Now apply the error bound for Taylor series with $a = 0$ and $M = e^d$ to get:

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1}, \ \forall |x| \leqslant d.$$

Taking the limit of the right hand side we get:

$$\begin{aligned}\lim_{n\to\infty}\frac{e^d}{(n+1)!}|x|^{n+1} &= e^d \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} \\ &= 0.\end{aligned}$$

It follows from the squeeze theorem that:

$$\lim_{n\to\infty}|R_n(x)| = 0,$$

and so:

$$\lim_{n\to\infty}R_n(x) = 0, \ \forall |x| \leqslant d.$$

Since d is arbitrary, the convergence is valid for all x.
Example 20.2.2. Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x, \ \forall x \in \mathbb{R}$.
Solution. First, we find the Maclaurin series. This is done by finding the form of the n-th term. We arrange the prerequisite computations for the first few terms of the sequence:

$$\begin{aligned}f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0\end{aligned}$$

Clearly, we see that the derivatives repeat in a cycle of four, therefore, we can write the Maclaurin series as follows:

$$\begin{aligned}T_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.\end{aligned}$$

Next, we must show that $R_n(x)$ approaches 0 as $n \to \infty$. Since $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, we have:

$$|f^{(n+1)}(x)| \leqslant 1, \ \forall x.$$

Apply the error bound formula with $M = 1$, to deduce that:

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1!)}.$$

It can be shown that:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

so by the squeeze theorem, $\lim_{n\to\infty}|R_n(x)| = 0$ and therefore, $\lim_{n\to\infty} R_n(x) = 0, \forall x$ as required.

Example 20.2.3. Find the Maclaurin series for $\cos x$ and prove that it represents $\cos x$, $\forall x \in \mathbb{R}$.

Solution. We could proceed directly as in the preceding example, but it is easier to differentiate the Maclaurin series for $\sin x$:

$$\begin{aligned}\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}[\sin x] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\end{aligned}$$

Since the Maclaurin series for $\sin x$ converges for all x, Theorem 34 tells us that the differentiated series for $\cos x$ also converges for all x. Thus:

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

Lecture No. 21

The Binomial Series

21.1 Using the Binomial Theorem

We can find the Maclaurin series for the function $f(x) = (1+x)^k$ where k is a real number. First, we consider expanding the general expression $(a+b)^n$, where n is some positive integer. The **Binomial Theorem** states that for any positive integer n:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i,$$

where $\binom{n}{i}$ is the **binomial coefficient** defined by:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}.$$

Example 21.1.1. For $n = 3$, we have:

$$(a+b)^3 = \sum_{i=0}^{3} \binom{3}{i} a^{3-i} b^i = a^3 + 3a^2b + 3ab^2 + b^3.$$

This theorem is used exclusively in the case where n in $(a+b)^n$ is strictly a positive integer. When n is not a positive integer, the theorem's conditions are not satisfied, however, the theorem is still useable. First, we take a look at the Maclaurin series for $(1+x)^k$ where k is any real number. For notational convenience, we extend the definition of the binomial coefficient to any real number k:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

By convention, $\binom{k}{0} = 1$.

Example 21.1.2. Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution. We first arrange the first few computations to deduce the general formula:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f^{(2)}(x) &= k(k-1)(1+x)^{k-2} & f^{(2)}(0) &= k(k-1) \\
f^{(3)}(x) &= k(k-1)(k-2)(1+x)^{k-3} & f^{(3)}(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Therefore, the Macluarin series of $f(x) = (1+x)^k$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

Which is known as the **Binomial Series**.

Note. If k is a positive integer, then eventually, the terms in the binomial series will be 0. For other values of k, none of the terms will be 0, and so we can try the ratio test to determine its convergence:

Let $a_n = \displaystyle\binom{k}{n} x^n$. Then:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right| \\
&= \frac{|k-n|}{n+1}|x| \\
&= \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x|
\end{aligned}$$

Taking the limits of both sides we see that:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

Thus by the ratio test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

The following theorem states that the function $(1+x)^k$ is indeed equal to its Maclaurin series.

Theorem 39. The Binomial Series. If k is any real number and $|x| < 1$, then:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

Example 21.1.3. Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$.

Solution. We rewrite $f(x)$ in a form where we can apply the binomial series:

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} \\ &= \frac{1}{2\sqrt{1-\frac{x}{4}}} \\ &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}\end{aligned}$$

Use the binomial series with $k=-\frac{1}{2}$, and with x replaced by $-\dfrac{x}{4}$:

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{x^n}{4^n} \\ &= \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{x^n}{4^n}\end{aligned}$$

We expand $\binom{-\frac{1}{2}}{n}$, for $n \geqslant 1$, as follows:

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} \\ &= (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\end{aligned}$$

Substituting this back into equation, we get:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{8^n n!}x^n.$$

From the binomial series theorem, we know that htis series converges when:

$$\left|-\frac{x}{4}\right| < 1,$$

that is, $|x|<4$, so the radius of convergence in this case is $R=4$.

Lecture No. 22

Finding Limits using Power Series

22.1 Using known Power Series

We can use power series to evaluate the limit of a function. Here are some important Maclaurin series:

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n!)} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$R = 1$

Example 22.1.1. Find the sum of the series:

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$$

Solution. Using sigma notation, we can write the series as:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}.$$

From the table above, this series matches the entry for $\ln(1+x)$ with $x = \frac{1}{2}$. So:

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{n \cdot 2^n} = \ln\left(1+\frac{1}{2}\right) = \ln\frac{3}{2}.$$