MH1200 Linear Algebra I.

Problem list for Week #2.

This week's topics:

- Systems of linear equations and their augmented matrices.
- Row Echelon form.
- Back-substitution.
- Gaussian elimination.

(I revised this set to add some extra problems. In case students have worked on this sheet already the new problems are indicated by letters A, B so on.)

Core problems:

Problem 1:

Write down the augmented matrix for each of the following systems of linear equations.

$$(a)\quad \begin{aligned} 7x_1 + x_2 + 4x_3 &= -3 \\ 3x_1 - 2x_3 &= 5 \\ -2x_2 + x_3 &= 7 \end{aligned} \qquad (b)\quad \begin{aligned} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

$$(c)\quad \begin{aligned} 2x_1 &= 0 \\ 3x_1 - 4x_2 &= 0 \\ x_2 &= 1 \end{aligned} \qquad (d)\quad \begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 3 \\ x_4 &= 4 \end{aligned}$$

Problem 2:

Which of the following augmented matrices are in row-echelon form.

$$(a)\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \qquad (b)\ \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad (c)\ \left(\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Problem 3:

Consider the linear system:

$$\begin{aligned} x + y + 2z &= a \\ x + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

Show that for this system to be consistent, the constants a, b, c must satisfy $a + b = c$. (Hint: Recall that "consistent" means "has a solution". Assume a solution exists and then try and show that the equation $a + b = c$ follows from these equations, which the solution satisfies.)

Problem 4:

Solve the following linear system by Gaussian elimination and back-substitution.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Problem 5:

Solve the following linear system by Gaussian elimination and back-substitution.

$$\begin{aligned} 2x_1 - 3x_2 &= -2 \\ 2x_1 + x_2 &= 1 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$

Problem A: (From a 2019 quiz.)

Consider the following system of linear equations in three variables x_1, x_2 and x_3:

$$\begin{aligned} x_1 + 0x_2 - 5x_3 &= 1 \\ 0x_1 + 10x_2 + 2x_3 &= 7 \end{aligned}$$

(i) Write down the augmented matrix for this system.

(ii) Is this matrix in row echelon form?

(iii) Give a parametrization for the set of solutions of this equation.

(iv) Briefly describe a geometric interpretation for this set of solutions. (But there is no need to draw a diagram.)

Problem B: (From the 2019 midterm.)

A quadratic function is a function of the form

$$y = ax^2 + bx + c$$

where $a \neq 0$, b, and c are real constants.

There are three parts below. In each part, determine how many quadratic functions there are which have graphs going through all of the points on the given list. The first entry of each pair (x, y) is the x-coordinate of the point and the second entry is its y-coordinate. Give details of your calculations.

(i) $(-1,-2),(0,1)$.

(ii) $(-1,-2),(0,1),(1,2)$.

(iii) $(-1,-2),(0,1),(1,2),(2,1)$.

Problem 6:

Let P be a plane in three dimensional space determined by the points $(1,0,1),(1,1,0)$ and $(0,1,1)$. Find a linear equation in x, y, z with this plane as its solution set.

Problem C. (From the final exam 2019.)

Give an example of a system of linear equations which simultaneously satisfies all of the following conditions:

- There are 2 variables.
- Four different equations.
- Every entry in the augmented matrix is non-zero.
- There is exactly 1 solution.

Justify your answer by calculating the set of solutions of your example. Show your working.

More abstract or challenging problems:

Problem 7:

Let L be the line in 3-dimensional space given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t-6 \\ t-10 \\ -2t+20 \end{pmatrix} \qquad t \in \mathbb{R}$$

Find a linear system that has L as its solution set.

Problem 8:

The sum of the cubes of the first n natural numbers

$$C(n) = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3 = \sum_{i=1}^{n} i^3$$

is expressible as a degree four polynomial in n:

$$C(n) = a_1 \cdot n + a_2 \cdot n^2 + a_3 \cdot n^3 + a_4 \cdot n^4 \ .$$

Set up a system of four linear equations that $a_1, \ldots, a_4$ must satisfy and solve for them. Bonus: prove that your answer actually is the formula for the sum of the first n cubes!

Problem 9:

In lectures we defined the concept of "row equivalence" which relates two matrices with the same shape (that is, the same number of rows and columns). We declared that two matrices with the same shape are row equivalent if there is a sequence of elementary row operations which takes you from one matrix to the other matrix. Explain why this does indeed define an equivalence relation on the set of matrices.

Problem 10:

Prove that it is impossible for a system of linear equations to have more than one but finitely many solutions. Hint: show that whenever you have two different solutions of a linear system, it is always possible to combine them somehow to construct new solutions.