MH1200 Linear Algebra I.

Problem set for Week #3.

This week's topics:

- Gauss-Jordan elimination.
- Counting solutions to systems of linear equations containing unknown constants.

Core problems:

Problem 1:

Solve the following linear system by *Gauss-Jordan elimination* and back-substitution.

$$\begin{aligned} 10y - 4z + w &= 1 \\ x + 4y - z + w &= 2 \\ 3x + 2y + z + 2w &= 5 \\ -2x - 8y + 2z - 2w &= -4 \\ x - 6y + 3z &= 1 \end{aligned}$$

Problem 2:

Solve the following linear system by *Gauss-Jordan elimination* and back-substitution.

$$\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}$$

Problem 3:

We consider the following linear system.

$$\begin{aligned} x + y + az &= 0 \\ x + ay + z &= 0 \\ ax + y + z &= 0 \end{aligned}$$

Here a is a constant. Find the values of a for which the system has
(i) no solution
(ii) exactly one solution
(iii) infinitely many solutions.

Problem 4:

We consider the following linear system.

$$\begin{aligned} ax + 9y + 2z &= 2 \\ x + ay + 4z &= 4 \\ ay + 2z &= 0 \end{aligned}$$

Here a is a constant. Find the values of a for which the system has
(i) no solution,
(ii) exactly one solution,
(iii) infinitely many solutions.

Problem 5: (From a Quiz in 2019.)

Determine the number of solutions of the following linear system in the variables x, y and z, depending on the value of an unknown constant a.

$$\begin{aligned} x + 3y + (2a^4 - 32)z &= 37 + 2a \\ 4y &= 28 \\ 2x + 6y + (5a^4 - 80)z &= 76 + 5a \end{aligned}$$

Problem 6:

Consider the system

$$\begin{aligned} ax + by = f \\ cx + dy = g \end{aligned} \ .$$

Assume that $a \neq 0$. Give a condition on a, b, c, d, f, g for this system to have a unique solution.

Problem 7:

Find an example of a linear system with 3 variables and 4 equations that has
(i) no solution,
(ii) exactly one solution,
(iii) infinitely many solutions with *two arbitrary parameters.*

Problem 8:

A certain physical quantity f depends on another quantity x by

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

where a, b, c, d, e are unknown constants. To determine a, b, c, d, e the following measurements have been taken.

x	0	1	2	3	4
$f(x)$	1	1	3	31	133

Find the values of a, b, c, d, e.

Problem 9:

Solve the following system of nonlinear (!) equations.

$$\begin{aligned} x^2 - y^2 + 2z^2 &= 6 \\ 2x^2 + 2y^2 - 5z^2 &= 3 \\ 2x^2 + 5y^2 + z^2 &= 9 \end{aligned}$$

More abstract or challenging problems:

Problem 10:

[Determining temperatures] Imagine we have a metal plate and that we fix the temperature at the boundaries of this plate (see picture) and let the plate come to a thermal equilibrium. The temperature of a point in the interior of the plate can be approximated as the average of the temperature of the 4 neighboring points: above, below, left, and right. Set up and solve the linear equations giving the temperatures T_1, T_2, T_3, T_4 in the interior of the plate.