MH1200 Linear Algebra I.

Problem set #4.

This week's topics:

- The elementary theory of matrices.
- The algebra of matrices.

Core problems:

Problem 1:

Let $\mathbf{A} = (a_{ij})_{3\times 4}$, where $a_{ij} = 2i - 3j$, $\mathbf{B} = \mathbf{I}_4$, $\mathbf{C} = \mathbf{0}_{3\times 3}$,

$$\mathbf{D} = (d_{ij})_{4\times 3} \text{ where } d_{ij} = \begin{cases} -1 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

$$\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

Evaluate the following, whenever possible.

(a) $\mathbf{AD}$, (b) $\mathbf{DA} - 3\mathbf{B}$, (c) $\mathbf{D}^2$, (d) $\mathbf{E}^2 + \mathbf{C}^3$,

(e) $\mathbf{DE} + 2\mathbf{D}$, (f) $\mathbf{EA}$, (g) $\mathbf{DB}$, (h) $\mathbf{CF}$,

(i) $\mathbf{AG}$, (j) $\mathbf{FE}$, (k) $\mathbf{EF}$, (l) $\mathbf{CA}$,

(m) $\mathbf{E} - \mathbf{E}^T$, (n) $\mathbf{F} - \mathbf{F}^T$, (o) $\mathbf{GG}^T$, (p) $\mathbf{G}^T\mathbf{G}$.

Problem 2:

The symbol $\sum$ is used to denote the sum of a sequence of numbers. For example,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

$$\sum_{x=0}^{m} f(x) = f(0) + f(1) + \cdots + f(m),$$

$$\sum_{k=1}^{r} c_{ik} d_{kj} = c_{i1} d_{1j} + c_{i2} d_{2j} + \cdots + c_{ir} d_{rj}.$$

Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{ij})$ an $n \times m$ matrix, with $m, n \geq 5$.

1. Each of the following sums represents an entry of either $\mathbf{AB}$ or $\mathbf{BA}$. Determine which matrix product is involved and which entry of that product is represented in each case:

$$(i) \quad \sum_{k=1}^{n} a_{3k} b_{k4} \qquad (ii) \quad \sum_{p=1}^{n} a_{4p} b_{p1} \qquad (iii) \quad \sum_{q=1}^{m} a_{q2} b_{3q} \qquad (vi) \quad \sum_{x=1}^{m} b_{2x} a_{x5}$$

2. Use the symbol $\sum$ to express the following entries symbolically.
   (a) In $\mathbf{AB}$, the entry in the 3rd row and 2nd column.
   (b) In $\mathbf{BA}$, the entry in the 4th row and 1st column.

Problem 3:

Using summation notation, prove that matrix multiplication is distributive. That is, prove that the following equation is true whenever it is well-defined (i.e. when all the shapes of the matrices match up):

$$(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

and

$$\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}.$$

(Maybe just bother proving the first equation. The second is exactly the same.)

Problem 4:

Recall that the trace $\text{tr}(\mathbf{A})$ of a square matrix $\mathbf{A}$ is defined to be the sum of its diagonal entries.

1. Show that the following basic properties of trace hold for any square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, and any scalar $r \in \mathbb{R}$:
   (a) $\text{tr}(\mathbf{A}+\mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$
   (b) $\text{tr}(r\mathbf{A}) = r\,\text{tr}(\mathbf{A})$
2. Show that for any matrices $\mathbf{C}$ and $\mathbf{D}$ where $\mathbf{C}$ is $p \times q$ and $\mathbf{D}$ is $q \times p$ then
$$\text{tr}(\mathbf{CD}) = \text{tr}(\mathbf{DC}).$$
   This crucial property of trace is called *cyclic invariance.*
3. Exploit the properties of trace you proved in part 1 and 2 to prove the following fact: There are no square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order that satisfy
$$\mathbf{AB} - \mathbf{BA} = \mathbf{I}.$$
   Here $\mathbf{I}$ denotes the identity matrix of the same order as $\mathbf{A}$ and $\mathbf{B}$.

Problem 5:

Let $\mathbf{A}$ be any diagonal matrix of order n. Show that a matrix $\mathbf{B}$ with the property that $\mathbf{AB} = \mathbf{I}$ exists if and only if $a_{ii} \neq 0$ for all i.

Problem 6:

Let $\mathbf{I}_n$ denote the $n \times n$-identity matrix. In other words this is the matrix whose (i, j)-entry is
$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
Prove that for every $m \times n$ matrix $\mathbf{A}$
$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}.$$

This is a precise proof. You should also think of a more intuitive way of understanding this, perhaps with the "dividing-up-into-blocks" trick for understanding matrix multiplication.

More abstract or challenging problems:

Problem 7: (Part (ii) appeared on 2018 Midterm.)

(i) Let $\mathbf{A}$ and $\mathbf{B}$ be matrices such that the matrix product $\mathbf{AB}$ is well-defined. Explain why the matrix product $\mathbf{B}^T\mathbf{A}^T$ is also well-defined and prove that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$$

(ii) Consider a square matrix $\mathbf{A}$. Show that $\mathbf{A}$ is invertible if and only if its transpose is invertible.

Problem 8:

In lectures we met the concept of a **lower triangular** matrix. This is a square matrix satisfying the property that the entries a_{ij} where $j > i$ are all zero. (In other words, the entries lying strictly above the diagonal are all zero.)

Prove that if both $\mathbf{A}$ and $\mathbf{B}$ are lower triangular matrices of the same order (i.e. the same square shape), then so is their product $\mathbf{AB}$.

Use transpose to immediately deduce a similar property for upper triangular matrices.

Problem 9:

A square matrix $\mathbf{A}$ is said to be

- an involutory matrix if $\mathbf{A}^2 = \mathbf{I}$,
- an idempotent if $\mathbf{A}^2 = \mathbf{A}$.

Show that every involutory matrix can be expressed as a difference of two idempotents.

Problem 10:

This problem concerns the Fibonacci sequence. The Fibonacci sequence is the sequence of numbers

$$F_0, F_1, F_2, \ldots$$

defined by setting $F_0 = 0$, $F_1 = 1$ and for every other $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. In other words, the nth Fibonacci number, for $n \geq 2$, is defined to be the sum of the two preceding Fibonacci numbers:

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \ldots.$$

In this problem we'll prove a beautiful historical fact about what the Fibonacci numbers look like when they are very large - to be precise, that

$$\lim_{n\to\infty} \left(F_n - \frac{1}{\sqrt{5}} \phi_+^n \right) = 0$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the classical *golden ratio*. (For example see the Wikipedia page for the golden ratio for a fascinating survey of countless places this number appears in art and nature).

The first step is to prove (probably using induction) that for all $k \in \mathbb{N}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}.$$

The next step is to prove that we can *diagonalize* this matrix via the following formula:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = P \begin{bmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{bmatrix} P^{-1}$$

where

$$\phi_\pm = \frac{1 \pm \sqrt{5}}{2}$$

and

$$P = \begin{bmatrix} 1 & 1 \\ \phi_+ & \phi_- \end{bmatrix}.$$

Note in passing that the golden ratio ϕ_+ and ϕ_- are the two roots of the equation $x^2 - x - 1 = 0$. Also know that "diagonalization" in general is a big topic in Linear Algebra II.

With these formulas you can write down a closed expression for F_n in terms of ϕ_+ and ϕ_- and from there you can conclude the limit.