MH1200 Linear Algebra I.

Problem set #5.

This week's topics:

- The theory of elementary matrices.
- Invertibility.

Comment: Some of the problems in this set can be solved using determinants. The theory of determinants is based on this week's material and for this week please use the more fundamental techniques as discussed in this week's lectures.

Core problems:

Problem 1:

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 1 & 1 & 0 & 4 \\ 0 & 4 & -1 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 2 & 2 & 0 & 8 \\ 0 & 4 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & -1 & 2 \\ 1 & 1 & 0 & 4 \\ 2 & 0 & 1 & -3 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 1 & 1 & 0 & 4 \\ -1 & 3 & -1 & -2 \end{pmatrix}.$$

Note that

$$\mathbf{A} \xrightarrow{R_2 \to 2R_2} \mathbf{B}, \qquad \mathbf{A} \xrightarrow{R_1 \leftrightarrow R_3} \mathbf{C}, \qquad \mathbf{A} \xrightarrow{R_3 \to R_3 - R_2} \mathbf{D}.$$

1. If $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are elementary matrices such that $\mathbf{E}_1\mathbf{A} = \mathbf{B}$, $\mathbf{E}_2\mathbf{A} = \mathbf{C}$, $\mathbf{E}_3\mathbf{A} = \mathbf{D}$, write down the matrices $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. Check your answers.
2. Find elementary matrices $\mathbf{E}_4$, $\mathbf{E}_5$ and $\mathbf{E}_6$ such that $\mathbf{E}_4\mathbf{B} = \mathbf{A}$, $\mathbf{E}_5\mathbf{C} = \mathbf{A}$ and $\mathbf{E}_6\mathbf{D} = \mathbf{A}$. What elementary row operations do these three elementary matrices correspond to?
3. Verify that $\mathbf{E}_1\mathbf{E}_4 = \mathbf{E}_2\mathbf{E}_5 = \mathbf{E}_3\mathbf{E}_6 = \mathbf{I}$.

Problem 2:

Let $\mathbf{A}$ and $\mathbf{B}$ be $4 \times n$ matrices such that $\mathbf{E}_1\mathbf{E}_2\mathbf{A} = \mathbf{E}_3\mathbf{E}_4\mathbf{B}$ where

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Describe how $\mathbf{B}$ can be obtained from $\mathbf{A}$ by elementary row operations.

Problem 3: (From the 2019 Midterm exam.)

Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 2 & -2 \\ -1 & 4 & 1 \end{bmatrix}.$$

(i) Give a sequence of elementary row operations which will transform this matrix into a matrix $\mathbf{B}$ in reduced row echelon form.

(ii) Find elementary matrices $\mathbf{E}_1, \ldots, \mathbf{E}_k$ such that

$$\mathbf{A} = \mathbf{E}_k \ldots \mathbf{E}_1 \mathbf{B}.$$

(iii) Using the information you deduced above, decide whether $\mathbf{A}$ is invertible or not. Briefly explain your deduction referring to any standard facts proved in the lectures you use.

(iv) Let $\mathbf{C}$ and $\mathbf{D}$ be matrices with the property that they are row equivalent to the same row reduced echelon form. Do there exist elementary matrices $\mathbf{E}'_j$ such that

$$\mathbf{C} = \mathbf{E}'_m \ldots \mathbf{E}'_1 \mathbf{D} \quad ?$$

Briefly justify.

Problem 4

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ a & -1 & -2 \\ 2 & 3 & 7 \end{pmatrix}$ where a is an unknown constant.

1. Find the values of a for which $\mathbf{A}$ is invertible.

   (Hint: One of the 4 characterisations of invertibility is that the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution. As always you can see if that is true by doing Gaussian elimination and seeing which values of a result in 1 solution.)

2. In the case of a such that $\mathbf{A}$ is invertible apply Gauss–Jordan Elimination to the 3×6 matrix $(\mathbf{A} \,|\, \mathbf{I})$ to determine its reduced row-echelon form. Use this to determine $\mathbf{A}^{-1}$. Check explicitly that your answer satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Problem 5

Let $\mathbf{A} = \begin{pmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{pmatrix}$.

Determine the values of a, b, c for which $\mathbf{A}$ is invertible and under those conditions find $\mathbf{A}^{-1}$.

Problem 6

Give conditions on a, b, c, d for the following matrix to be invertible

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

(There is no need to find the inverse in this case.)

Problem 7

Let $\mathbf{U}$ be any matrix satisfying the two properties that

- $\mathbf{U}$ is upper triangular
- every element on the diagonal is non-zero.

Briefly explain why $\mathbf{U}$ is invertible.

More abstract or challenging problems:

Problem 8

Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the inverse of this matrix?

Now extend $\mathbf{A}$ to a general $n \times n$ matrix $\mathbf{A}_n$ by continuing this alternating pattern of 1's and -1's. Can you write down the inverse of the general case ${\mathbf{A}_n}^{-1}$? Give some justification. (The case $n = 8$ is illustrated in the solutions.)

Problem 9

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same order. Prove that if $\mathbf{A}$ is singular (i.e. not invertible) then $\mathbf{AB}$ and $\mathbf{BA}$ are also both singular.

Hint: Go back and have a look at our 4 characterisations of invertibility. If you use one of these 4 characterisations in the right way you can give a simple proof of this property.

Problem 10

Consider a 4×4 matrix $\mathbf{A}$ where each column has exactly the numbers 0, 1, 2, and -3 in some unspecified order. Is it possible for such a matrix to be invertible? Justify your answer.