MH1200 Linear Algebra I.

Problem set #6.

This week's topics:

- Definition of Determinant.
- Elementary properties of determinants.

Core problems:

Problem 1:

For each of the following matrices, compute the determinant using cofactor expansion along some row or some column.

$$(a)\quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad (b)\quad \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix},$$

$$(c)\quad \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad (d)\quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Problem 2:

For each of the following matrices, find the values of λ for which $\det(\mathbf{A}) = 0$.

$$(a)\quad \mathbf{A} = \begin{pmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{pmatrix}, \qquad (b)\quad \mathbf{A} = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{pmatrix}.$$

Problem 3:

Consider the following two matrices depending on 9 parameters $a, b, c, d, e, f, g, h, i$:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} g & h & i \\ 7d - 3g & 7e - 3h & 7f - 3i \\ a + 10d & b + 10e & c + 10f \end{bmatrix}$$

Using the 6 fundamental properties of determinants introduced in the lectures, compute $\det(\mathbf{B})$ in terms of $\det(\mathbf{A})$.

Problem 4:

Show the following calculation of a determinant which depends on parameters a, b, and c:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

State under which conditions on the parameters this matrix is invertible.

Hint: It is possible to do a direct computation of this using a cofactor expansion but it will be more revealing and elegant to manipulate this matrix using row operations. This determinant is called a *Vandermonde determinant*. In the challenging problems section we'll consider generalizing this to higher order examples.

Problem 5:

In this problem we'll think about whether the determinant respects the basic operations on matrices. We discussed the important fact that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ in lectures. But what about the other two basic operations?

1. Let $\mathbf{A}$ be a square $m \times m$ matrix and let $r \in \mathbb{R}$. Conjecture a relationship between $\det(r\mathbf{A})$ and $\det(\mathbf{A})$.
2. Use induction to formally prove the relationship you conjectured in Part (i).
3. Is the following formula true? Either prove it or give and example to show it cannot be true.

$$\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B}).$$

Problem 6:

Let $\mathbf{A}$ be a 4×4 matrix such that $\det(\mathbf{A}) = 9$. Find

(a) $\det(3\mathbf{A})$, (b) $\det(\mathbf{A}^{-1})$, (c) $\det(3\mathbf{A}^{-1})$, (d) $\det((3\mathbf{A})^{-1})$.

Problem 7:

Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{B}.$$

1. Describe how $\mathbf{A}$ can be obtained from $\mathbf{B}$ by elementary row operations.
2. If $\det(\mathbf{A}) = 4$, find $\det(\mathbf{B})$.

Problem 8: (From final exam 2018.)

Consider the following matrix, where $a_1, \ldots, a_{16}$ are real constants:

$$\begin{bmatrix} 0 & a_1 & 0 & a_2 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & a_8 & 0 & a_9 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{15} & 0 & a_{16} & 0 \end{bmatrix}.$$

What is its determinant? Briefly justify your answer.

More abstract or challenging problems:

Problem 9:

Let $\mathbf{A}$ be a 3×3 matrix whose entries are all either 0 or 1. What is the largest value and smallest value of $\det(\mathbf{A})$ (among all 3×3 matrices with entries being 0 or 1 only)?

Hint: write down a polynomial which equals the determinant of a general 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and think about what possible values that expression can take when the parameters $a, \ldots, i$ are all zero or one.

Problem 10:

Consider the following 4×4 *Vandermonde matrix* depending on variables a, b, c, d:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

Show that $\det(\mathbf{A}) = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$. So under which conditions on a, b, c, d is $\mathbf{A}$ invertible?

Problem 11:

Generalize the identity in the previous problem to general $n \times n$ Vandermonde matrices.

Comment: writing down a complete and correct proof by induction of the general identity is very long, so in this general case it may not be worth trying to write down a complete proof. Just work out what the answer should be and why it should be true. There is also an elegant proof of this using algebraic ideas, but that is better to see when you have learnt some more advanced algebraic ideas such as you will learn in Abstract Algebra I.

Problem 12: (From the 2019 midterm.)

(i) Consider the following matrix depending on parameters $a, \dots, l \in \mathbb{R}$.

$$\mathbf{A} = \begin{bmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{bmatrix}.$$

Show that $\det(\mathbf{A}) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \det\left(\begin{bmatrix} i & j \\ k & l \end{bmatrix}\right)$.

(ii) More generally, consider the following matrix. Here $\mathbf{X}$ is a square $l \times l$ matrix and $\mathbf{Z}$ is a square $k \times k$ matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0}_{k \times l} & \mathbf{Z} \end{bmatrix}.$$

Note that $\mathbf{B}$ has a block of zeroes in the bottom-left corner. A matrix of this form is called "block upper triangular". Show that for such a matrix:

$$\det(\mathbf{B}) = \det(\mathbf{X}) \det(\mathbf{Z}).$$

Note that for this part of the problem, full marks will only be given for a completely clear and correct proof.

Problem 13:

Prove that the two definitions of determinant we introduced in lectures:

- via a cofactor expansion
- via a sum over permutations

are equal.