MH1200 Linear Algebra I.

Problem set #7.

This week's topics:

- The two big theorems about determinants.
- The adjoint matrix of a matrix.
- Cramer's rule.

Core problems:

Problem 1:

Compute the inverses of the following matrices by using adjoint matrices.

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ -1 & 6 & 2 \end{pmatrix}.$$

Problem 2:

Let $\mathbf{A}$ be an invertible square matrix all of whose entries are integers. Show that all entries of $\mathbf{A}^{-1}$ are integers if and only if $\det(\mathbf{A}) = \pm 1$.

Comment: Please observe that because I wrote "if and only if" there are two logical directions in this statement you have to explain.

Problem 3:

Solve the following linear systems by applying Cramer's rule.

(i)

$$\begin{aligned} x + y - z &= 2 \\ 3x - y + z &= 5 \\ 3x + 2y + 4z &= 0 \end{aligned}$$

(ii)

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 4 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 - x_4 &= -2 \\ -x_1 + x_2 - x_3 + x_4 &= 4 \end{aligned}$$

Problem 4:

Let

$$\mathbf{A} = \begin{pmatrix} 8 & a & 0 \\ 0 & 8 & a \\ a & 0 & 8 \end{pmatrix}$$

where a is a real number.

(i) Compute $\det(\mathbf{A})$. For what values of a is $\mathbf{A}$ invertible?

(ii) Under the condition that $\mathbf{A}$ is invertible, solve the matrix equation

$$\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

where b, c, d are real numbers. (Hint: The quickest way to do this now we have calculated the determinant of $\mathbf{A}$ is using Cramer's Rule.)

Problem 5:

Let

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix},$$

where a and b are real numbers.

(i) Evaluate $\det(\mathbf{A})$. For what values of a and b is $\mathbf{A}$ invertible?

(ii) Compute the adjoint matrix of $\mathbf{A}$ for all values of a and b, and find $\mathbf{A}^{-1}$ for all values of a and b for which $\mathbf{A}$ is invertible.

(iii) Consider the matrix equation

$$\mathbf{AXA} = \mathbf{I},$$

where $\mathbf{I}$ is an identity matrix. Determine the values of a and b for which a solution $\mathbf{X}$ exists (justify your answer), and solve the matrix equation in these cases.

Problem 6: (A Quiz problem from 2019.)

Consider the following matrix $\mathbf{A}$, where a is a constant real number.

$$A = \begin{bmatrix} 1 & 1+a & 0 \\ 0 & 2 & 0 \\ a & a & 1 \end{bmatrix}.$$

(a) For what values of the constant a is A invertible?

(b) Calculate $\operatorname{adj}(A)$, the adjoint of A.

(c) Use part (b) to solve the equation

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

More abstract or challenging problems:

Problem 7:

Let $\mathbf{A}$ be a n-by-n matrix. Let $\mathbf{B}$ be the matrix where the (i,j) entry $\mathbf{B}(i,j)$ is determined by the corresponding (i,j) entry of $\mathbf{A}$ via the formula $\mathbf{B}(i,j) = \frac{i}{j}\mathbf{A}(i,j)$. For example, in the 3-by-3 case, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{3}a_{13} \\ 2a_{21} & a_{22} & \frac{2}{3}a_{23} \\ 3a_{31} & \frac{3}{2}a_{32} & a_{33} \end{bmatrix}.$$

How does the determinant of $\mathbf{B}$ relate to that of $\mathbf{A}$?

Comment: There are many ways to approach this. One elegant way is to find a matrix equation that expresses $\mathbf{B}$ in terms of $\mathbf{A}$.

Problem 8:

Let $\mathbf{B}$ be an arbitrary $m \times n$ matrix.

(i) Show that the matrix $\mathbf{B}^T\mathbf{B}$ is a symmetric matrix.

(ii) Is it possible for every symmetric matrix $\mathbf{A}$ to be written as $\mathbf{B}^T\mathbf{B}$ for some matrix $\mathbf{B}$ of the same shape as $\mathbf{A}$? Justify your answer.

Problem 9: (Proof of Cramer's Rule.)

In this exercise we will discover an easy proof of Cramer's rule. Let $\mathbf{A}$ be an invertible $n \times n$ matrix, and let $\mathbf{B}$ be a $n \times 1$ matrix.

Because $\mathbf{A}$ is invertible the matrix equation $\mathbf{AX} = \mathbf{B}$ has a unique solution $\mathbf{X}$. Denote the components of the unique solution $\mathbf{X}$ via

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T.$$

(a) For $1 \le i \le n$ let $\mathbf{C}_i$ denote the $n \times 1$ matrix obtained from the i-th column of $\mathbf{A}$. Show that

$$\mathbf{B} = x_1\mathbf{C}_1 + x_2\mathbf{C}_2 + \ldots + x_n\mathbf{C}_n.$$

(There are various ways to see this. One interesting way is using the "splitting a matrix product up into blocks" trick.)

(b) Let $\mathbf{A}_j$ denote the matrix obtained from $\mathbf{A}$ by replacing the j-th column with the matrix $\mathbf{B}$. Use fundamental properties of determinants to prove that

$$\det(\mathbf{A}_j) = x_j \det(\mathbf{A}).$$

(c) Deduce Cramer's rule.

Problem 10: (From final exam 2019.)

(a) Let $\mathbf{A}$ be an $n \times n$ matrix with the property that $\mathbf{A}^m = \mathbf{0}_{n\times n}$ where m, n are certain integers greater than zero. Calculate the determinant of $\mathbf{A}$. Justify your answer.

(b) Let $\mathbf{B}$ be an arbitrary $m \times m$ matrix for some integer m greater than zero. Carefully derive a formula for $\det(\mathrm{adj}(\mathbf{B}))$ in terms of $\det(\mathbf{B})$.

Problem 11:

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices with determinant 1 and let $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

For each k such that $1 \le k < n$ let $\mathbf{X}_k$ denote the $(n-k) \times (n-k)$ matrix that results from forgetting the first k rows and the last k columns of $\mathbf{X}$. (In words, $\mathbf{X}_k$ is the lower-left square $(n-k) \times (n-k)$ corner block appearing inside the matrix $\mathbf{X}$. It will probably be helpful to draw this on a diagram of the matrix.)

Let $\mathbf{Y}_k$ be the $n \times n$ matrix obtained by making the first k columns of $\mathbf{Y}_k$ the first k columns of $\mathbf{A}$, and making the remaining $(n-k)$ columns of $\mathbf{Y}$ the first $(n-k)$ columns of $\mathbf{B}$.

Prove that for all $1 \le k < n$:

$$\det(\mathbf{X}_k) = \det(\mathbf{Y}_k)$$