MH1200 Linear Algebra I.

Problem set #8.

This week's topics:

- The definition of a subspace of $\mathbb{R}^n$.
- The basic theory of subspaces.

Core problems:

Problem 1:

Determine which of the following subsets of $\mathbb{R}^3$ or $\mathbb{R}^4$ are actually subspaces. Give a brief justification. Reminder: To show a subset is in fact a subspace you either have to directly check that the axioms are true, or you may use the fact we proved in lectures that the set of solutions to a homogeneous system of linear equations is always a subspace. Another possibility is to show that the subset you are studying is the span of a set of vectors, which we also proved is always a subspace. On the other hand, to show that a subset isn't a subspace, it is enough to exhibit one specific example where one of the axioms fails.

Comment: if you are struggling with set theory notation, a good way to start this problem is to write out a few vectors in each of the sets below so you understand what the sets are first. You take numbers satisfying the conditions on the right of the | symbol, then substitute them into the formulae to the left of the | symbol.

1. $\{ (0,0,0) \}$.
2. $\{ (1,1,1) \}$.
3. $\{ (0,0,0),\ (1,1,1) \}$.
4. $\{ (0,0,c); c \in \mathbb{Z} \}$.
5. $\{ (0,0,c); c \in \mathbb{R},\ c \geq 0 \}$.
6. $\{ (0,0,c); c \in \mathbb{R} \}$.
7. $\{ (1,1,c); c \in \mathbb{R} \}$.
8. $\{ (a,b,c); a,\ b,\ c \in \mathbb{R} \text{ and } a \geq b \geq c \}$.
9. $\{ (a,b,c); a,\ b, c \in \mathbb{R} \text{ and } 4a = 3b \}$.
10. $\{ (a,b,b); a,\ b \in \mathbb{R} \}$.
11. $\{ (a,b,ab); a, b \in \mathbb{R} \}$.
12. $\{ (a^2,b^2,c^2); a, b, c \in \mathbb{R} \}$.

13. $\{ (a^3, b^3, c^3); a, b, c \in \mathbb{R} \}$.

14. $\{ (x, y, z); x, y, z \in \mathbb{R} \quad \text{and} \quad x + y - 2z = 0\}$.

15. $\{ (x - 1, y, z); x, y, z \in \mathbb{R} \quad \text{and} \quad x + y - 2z = 0\}$.

16. $\{ (x, y - 1, z); x, y, z \in \mathbb{R} \quad \text{and} \quad x + y - 2z = 1\}$.

17. $\{ (x_1, x_2, x_3, x_4); x_1, x_2, x_3, x_4 \in \mathbb{R} \quad \text{and} \quad x_1 = x_2 + x_3 = x_4 - x_3 + 2x_1\}$.

Problem 2:

Let U, V, W be three planes in $\mathbb{R}^3$ where

$$U = \{ (x, y, z); 2x - y + 3z = 0 \},\ V = \{ (x, y, z); 3x + 2y - z = 0 \},$$
$$W = \{ (x, y, z); x - 3y - 2z = 1 \}.$$

1. Determine which of U, V, W contain the origin.
2. Give parameterizations for the sets $U \cap V$ and $V \cap W$. (In other words: find a parameterization for the general solution of the corresponding linear system which describes the intersection).
3. Is $U \cap V$ a subspace of $\mathbb{R}^3$? Is $V \cap W$ a subspace of $\mathbb{R}^3$? Justify your answers.

Problem 3:

Let V be a subspace of $\mathbb{R}^3$. Define a corresponding subset W of $\mathbb{R}^2$ by $W = \{(x, z); (x, y, z) \in V\}$. (In words, W is the subset of $\mathbb{R}^2$ consisting of pairs of numbers with the property that the pair is the first coordinate and the third coordinate of some vector in the subspace V.)

(i) Show that W is a subspace of $\mathbb{R}^2$.

(ii) What is W if $V = \{(x, y, 0); x, y \in \mathbb{R}\}$?

Problem 4: (Assembled from different versions of a Quiz in 2019.)

Consider the following subsets of $\mathbb{R}^3$. In each case, state whether the subset is or is not a subspace of $\mathbb{R}^3$. In each case justify your answer.

(i) $S_1 = \{(a,b,c) \in \mathbb{R}^3; a+b=0, b+c+1=0\}$.

(ii) $S_2 = \left\{ (a,b,c) \in \mathbb{R}^3; \begin{vmatrix} 1 & 2 & 3 & 4 \\ a & 0 & b & 0 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 14 \end{vmatrix} = 0 \right\}$.

(iii) $S_3 = \left\{ (a,b,c) \in \mathbb{R}^3; \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b & c \\ 2 & 1 & 1 \end{vmatrix} = 0 \right\}$.

(iv) $S_4 = \{(a,b,c) \in \mathbb{R}^3; a+b=0, b+c=0\}$.

(v) $S_5 = \left\{ (a,b,c) \in \mathbb{R}^3; \det\left(\begin{bmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{bmatrix} \right) = 0 \right\}$.

(vi) $S_6 = \left\{ (a,b,c) \in \mathbb{R}^3 : \begin{vmatrix} 1 & 2 & 3 & 4 \\ a^2 & 0 & b^2 & 0 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{vmatrix} = 0 \right\}$.

Problem 5: (Appeared on the final exam 2018.)

(a) Write down the formula for the trace of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now consider the following subsets of $\mathbb{R}^4$. In each case either briefly prove the given set is a subspace of $\mathbb{R}^4$, or briefly prove it is not a subspace.

(b) $S_1 = \left\{ (a,b,c,d) \in \mathbb{R}^4; \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\}$.

(c) $S_2 = \left\{ (a,b,c,d) \in \mathbb{R}^4; \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \geq 0 \right\}$.

(d) $S_3 = \left\{ (a,b,c,d) \in \mathbb{R}^4; \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 \right) = 0 \right\}$.

(e) $S_4 = \left\{ (a,b,c,d) \in \mathbb{R}^4; \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \right) \geq 0 \right\}$.

Hint: Don't be distracted by these matrix expressions. This question is about subsets of $\mathbb{R}^4$. The simplest thing is to calculate these matrix expressions and just work with the expressions that result from the calculation like you did in Question 1.

Problem A **(From final exam 2019.)**

(a) Consider the following subset of $\mathbb{R}^3$

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1 x_2 x_3 = 0\right\}.$$

Either prove that it is a subspace of $\mathbb{R}^3$ or prove that it is not.

(b) Give an example of a non-empty subset U of $\mathbb{R}^2$ simultaneously satisfying the two properties that

- U is closed under vector addition
- whenever $\mathbf{u} \in U$ then $(-1)\mathbf{u} \in U$ as well

but which is not a subspace of $\mathbb{R}^2$.

More abstract or challenging problems:

Problem 6:

In this problem we'll interpret $\mathbb{R}^n$ to be the set of column vectors with n entries. In other words, instead of writing the list horizontally

$$(u_1, \ldots, u_n) \in \mathbb{R}^n$$

in this problem we'll write the list vertically

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

Now let $\mathbf{A}$ be an $m \times n$ matrix. Define $V = \{\, \mathbf{Au} \mid \mathbf{u} \in \mathbb{R}^n \,\} \subset \mathbb{R}^m$. In words: V is the set of all possible vectors you can get by left-multiplying $\mathbf{A}$ onto some column vector $\mathbf{u}$. This set is called the "image" of the matrix $\mathbf{A}$ in the codomain $\mathbb{R}^m$.

1. Show that V is a subspace of $\mathbb{R}^m$.
2. Give an explicit parametrization for the subspace V in the cases

$$\text{(i)}\quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}. \qquad \text{(ii)}\quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}.$$

Comment on the next two problems: Problems 7 and 8.

In the next two problems we'll consider whether there is a natural way to combine two subspaces V_1, V_2 of some $\mathbb{R}^n$ to create a new subspace of $\mathbb{R}^n$ which contains both V_1 and V_2. We know from a theorem proved in lectures that the intersection $V_1 \cap V_2$ is always another subspace, but usually it doesn't contain V_1 and V_2 so we need to guess again. We might guess that the union of the subspaces $V_1 \cup V_2$ would be the right thing to do, but in Problem 6 you'll show that the union is never a subspace except for the special cases where one of V_1, V_2 is contained in the other one. In Problem 7 you'll learn the correct way to combine two subspaces to make a new subspace: the **sum** of subspaces $V_1 + V_2$.

Problem 7:

Consider two subspaces V_1 and V_2 of some $\mathbb{R}^n$. Show that if their union $V_1 \cup V_2$ is also a subspace of $\mathbb{R}^n$ then either $V_1 \subset V_2$, or $V_2 \subset V_1$, (or both, in which case the subspaces are equal).

Problem 8:

Consider two subspaces V_1 and V_2 of some $\mathbb{R}^n$. Define the sum of these subspaces $V_1 + V_2$ to be the subset of $\mathbb{R}^n$ defined by the set theoretic expression

$$V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2; \vec{v}_1 \in V_1, \vec{v}_2 \in V_2\}.$$

In words: $V_1 + V_2$ is the set of all possible vectors in $\mathbb{R}^n$ you can get by taking a vector $\vec{v}_1$ from V_1, a vector $\vec{v}_2$ from V_2, and adding them together: $\vec{v}_1 + \vec{v}_2$.

1. Show that $V_1 + V_2$ is indeed a subspace of $\mathbb{R}^n$ containing both V_1 and V_2.
2. Moreover, explain why if W is *any* subspace of $\mathbb{R}^n$ containing both V_i then $V_1 + V_2 \subset W$.