MH1200 Linear Algebra I.

Problem set #9.

This week's topics:

- Span.

Core problems:

Problem 1:

Let V be the subset of $\mathbb{R}^4$ given by

$$V = \{\,(w,x,y,z) \mid w,x,y,z \in \mathbb{R}^4, \quad w+x = y+z = -x+2y\}.$$

1. Show that V is a subspace of $\mathbb{R}^4$.
2. Express V as the span of a collection of vectors in $\mathbb{R}^4$.

Problem 2:

Consider the homogeneous linear system

$$\begin{aligned} 10x_2 - 4x_3 + x_4 &= 0 \\ x_1 + 4x_2 - x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\ -2x_1 - 8x_2 + 2x_3 - 2x_4 &= 0 \\ x_1 - 6x_2 + 3x_3 &= 0 \end{aligned}$$

Express the solution set as the span of some vectors in $\mathbb{R}^4$.

Problem 3:

Consider the following collection of vectors in $\mathbb{R}^4$:

$$S = \{\,(0,1,0,0)\,,(-a,0,1,-1)\,,(0,a,0,0)\,,(1,0,0,1)\}$$

where a is a real number.

(i) Show that no matter what we set a to, $\operatorname{span}(S) \neq \mathbb{R}^4$.

(ii) Find an example of a vector in $\mathbb{R}^4$ that is not contained in $\operatorname{span}(S)$.

Problem 4:

Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{u_1} = (2, 0, 2, -4),\ \mathbf{u_2} = (1, 0, 2, 5),\ \mathbf{u_3} = (0, 3, 6, 9),\ \mathbf{u_4} = (1, 1, 2, -1),$$
$$\mathbf{v_1} = (-1, 2, 1, 0),\ \mathbf{v_2} = (3, 1, 4, 0),\ \mathbf{v_3} = (0, 1, 1, 3),\ \mathbf{v_4} = (-4, 3, -1, 6).$$

Determine if the following are true.

(i) $\text{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\} \subset \text{span}\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\}$.

(ii) $\text{span}\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\} \subset \text{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\}$.

(iii) $\text{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\} = \mathbb{R}^4$.

(iv) $\text{span}\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\} = \mathbb{R}^4$.

Problem 5:

Determine which of the following statements are true. Justify your answer.

(i) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u} \neq \mathbf{v}$, $\mathbf{u} \neq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$, then $\text{span}\{\mathbf{u}, \mathbf{v},\ \mathbf{w}\} = \mathbb{R}^3$.

(ii) If S_1 and S_2 are two subsets of a real vector space, then $\text{span}(S_1 \cap S_2) = \text{span}(S_1)\ \cap\ \text{span}(S_2)$.

(iii) If S_1 and S_2 are two subsets of a real vector space, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1)\ \cup\ \text{span}(S_2)$.

Problem 6:

Let V be the subset of $\mathbb{R}^4$ given by

$$V = \{\, (a, b, c, d) \mid a, b, c, d \in \mathbb{R} \quad \text{and} \quad a + b = c + d\}.$$

(i) Show that V is a subspace of $\mathbb{R}^4$.

(ii) Let $S = \{\, (1, 0, 2, -1),\ (0, -3, 1, -4),\ (3, 6, 4, 5)\}$.
Is $\text{span}(S) \subseteq V$? Is $V \subseteq \text{span}(S)$? Is $\text{span}(S) = V$?

More abstract or challenging problems:

Problem 7:

Consider some subspace $V \subset \mathbb{R}^n$ and assume that V is spanned by vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}.$$

(i) Show that the following is also a spanning set

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{m-1} - \mathbf{v}_m, \mathbf{v}_m\}.$$

(ii) Show that the following is also a spanning set

$$\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \ldots + \mathbf{v}_m\}.$$