MH1300 FOUNDATIONS OF MATHEMATICS

2020/21 Semester 1

Tutorial 9

Ex. 5.4.8. Suppose that $h_0, h_1, h_2, \cdots$ is a sequence defined as follows:

$$h_0 = 1, \; h_1 = 2, \; h_2 = 3,$$
$$h_k = h_{k-1} + h_{k-2} + h_{k-3}, \text{ for all integers } k \geq 3.$$

(a) Prove that $h_n \leq 3^n$ for all $n \geq 0$.
(b) Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

Ex. 5.4.13. Use Strong mathematical induction to prove the existence part of the unique factorization theorem (Theorem 4.3.5): Every integer greater than 1 is either a prime number or a product of prime numbers.

Ex. 5.4.18. Compute $9^0, 9^1, 9^2, 9^3, 9^4$ and 9^5. Make a conjecture about the units digit of 9^n where n is a positive integer. Use (strong) mathematical induction to prove your conjecture.

Ex. 5.4.26. Suppose $P(n)$ is a property such that

(1) $P(0), P(1)$ and $P(2)$ are all true.
(2) for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true. Must it follow that $P(n)$ is true for all integers $n \geq 0$? If yes, explain why; if no, give a counterexample.

Additional: If the answer to (2) above is no, make one *small* change so that $P(n)$ is true for all $n \geq 0$. (Hint: Change times to plus).

Ex. E1 (The tower of Hanoi) The "Tower of Hanoi" is a well-known mathematical puzzle where we have three rods and n disks of different diameters. At the beginning of the game all the n disks are stacked at the first rod, in order of size, with the largest disk being at the bottom of the stack. The objective of the puzzle is to move the entire stack to the third rod, obeying the following simple rules:

(1) Only one disk can be moved at a time.
(2) Each move consists of taking a topmost disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the topmost disk on a stack.
(3) No disk may be placed on top of a smaller disk.

See Figure 1 for an illustration of the puzzle with $n = 3$ disks. In this case, it can be solved in 7 moves (see Figure 2). Prove that for each $n \geq 1$, we can solve the puzzle using at most $2^n - 1$ moves. (This is also optimal, but we will not prove it).

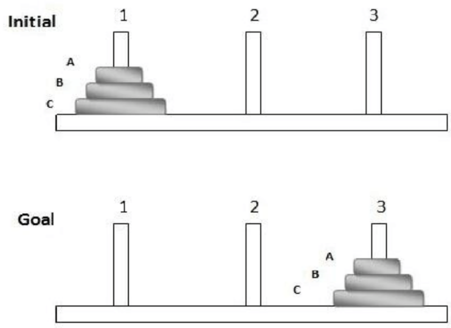

FIGURE 1. An illustration of the game with $n = 3$ disks

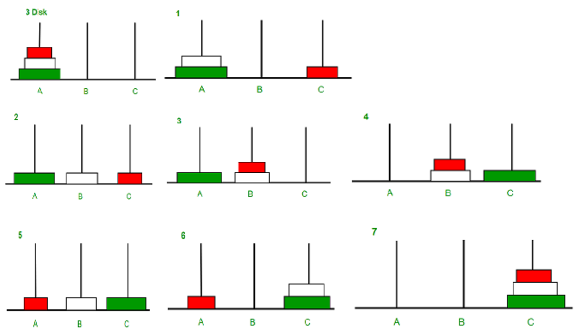

FIGURE 2. Solution for $n = 3$ in 7 moves

Ex. 4.8.16. Use the Euclidean algorithm to hand-calculate the greatest common divisors of the pair of integers

$$4131 \qquad \text{and} \qquad 2431.$$

Ex. 4.8.21. Complete the proof of the lemma in the lecture notes by proving the following: If a and b are any integers with $b \neq 0$ and q and r are any integers such that

$$a = bq + r,$$

then

$$\gcd(b, r) \leq \gcd(a, b).$$

Ex. 4.8.27.
Definition: The **least common multiple** of two nonzero integers a and b, denoted $\mathbf{lcm(a, b)}$, is the positive integer c such that

a. $a|c$ and $b|c$,
b. for all positive integers m, if $a|m$ and $b|m$, then $c \leq m$.

Prove that for all positive integers a and b, $a|b$ if and only if $\text{lcm}(a, b) = b$.

Additional Exercises On Complex Numbers

Ex. 1. Prove that for two complex numbers w and z, we have the triangle inequality $|z + w| \leq |z| + |w|$. Next, use it to prove $|z - w| \geq ||z| - |w||$.

Ex. 2. Find the 6th-roots of $5 - i5\sqrt{3}$.

Ex. 3. Using expressions for Re z and Im z, show that the hyperbola $x^2 - y^2 = 1$ can be written as

$$z^2 + \bar{z}^2 = 2.$$

Ex. 4. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.