MH1301 Bonus Question 3

Philip Thomas K.

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1. First we show that the following equality holds:

$$n - k \leqslant m$$

If we separate G into its k different sub-graphs, we have the graphs $G_1, G_2, \dots, G_k$ with $n_1, n_2, \dots, n_k$ vertices respectively. We know that the minimum number of edges required to connect n vertices is $n-1$. Thus, the lower bound for the total number of edges in G will have k sub graphs, each with the minimum number of edges required.

Note. $n_1 + n_2 + n_3 + \cdots + n_k = n$

Then if we add the number of edges in each minimised sub graph, we have:

$$\begin{aligned} \sum_{i=1}^{k}(n_i - 1) &= \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 \\ &= (n_1 + n_2 + \cdots + n_k) - k \\ &= n - k. \end{aligned}$$

Thus the minimum number of edges in G is $n - k$, so we have:

$$n - k \leqslant m.$$

2. To show that the following equality holds:

$$m \leqslant \frac{(n-k)(n-k+1)}{2}$$

Note that if we fix some arbitrary $n, k \in \mathbb{Z}$ and maximize the number of edges, we have a graph where there are $k-1$ sub graphs with only one vertex and one big sub-graph which is a complete graph.

Proof. If we consider that instead, we have k connected components, but of the k connected components, 2 graphs have more than 1 vertex. Let us call these graphs G_1, G_2 which have n_1, n_2 vertices respectively, where both sub graphs are complete graphs.

Then if we pick the graph with less vertices, which for this example we choose G_2, and take one of its vertices and remove it, we end up removing $n_2 - 1$ edges. Then, if we add the removed vertex to G_1, we add n_1 edges.

Since $n_2 - 1 < n_1$ we have increased the total number of edges in the graph. Therefore the maximum number of edges is achieved when exactly one component has more than one vertex. □

The big sub-graph will have exactly $n - k + 1$ vertices, since all the other components contain exactly one vertex. Since this graph is a complete graph we can obtain its number of edges:

$$\binom{n-k+1}{2} = \frac{(n-k)(n-k+1)}{2}.$$

Which is the upper limit for the number of edges in a graph with n vertices and k connected components, and therefore:

$$m \leqslant \frac{(n-k)(n-k+1)}{2}.$$

Which ultimately means that:

$$n - k \leqslant m \leqslant \frac{(n-k)(n-k+1)}{2}.$$

As required.